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Survey Article

Why we need Non absolute integral in place of Lebesgue integral?

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Abstract**:** In this survey note we discuss about non absolute integrable functions and we put our view about the question: Why we need Non absolute integral in place of Lebesgue integral? Various areas are discussed, where we can find Henstock-Kurzweil integral in place of Lebesgue integral.

Keywords**:** Henstock-Kurzweil integral; Lebesgue measure; Feynman path integral
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Survey

At the end of the seventeenth century Isaac Newton (1642-1727) and Gottfried Leibniz (1646-1716) independently discovered differential and integral calculus. Differential calculus was used to define the slope of a curve at a particular point and integral calculus was used to compute the area under a curve. As it turns out, for a large class of functions there was an inverse relationship between differential and integral calculus in the sense that a differentiable function could be obtained up to a constant by integrating its derivative. In fact, a function $F: [a, b] \to \mathbb{R}$ which is differentiable everywhere on $[a, b]$ except on a loosely stated small set $S \subseteq [a, b]$, could in certain cases be recovered by integrating a function $f: [a, b] \to \mathbb{R}$ which satisfies $F' = f$ on $[a, b] \setminus S$. Exactly how small S had to be and how well-behaved F had to be on S was not known at the time. The mathematics of that time simply did not allow mathematicians to conduct research into descriptive characterizations of integrable functions and consequently many aspects of integration theory remained shrouded in mystery. During the eighteenth century mathematicians began to realize that the very foundation of mathematical analysis was highly unstable and loosely defined. When geometric intuition no longer was sufficient, this served as a massive obstacle in research. Consequently, in the nineteenth century mathematicians began developing a rigorous framework for mathematical analysis using various epsilon-delta type definitions and proofs. In particular, it was the formal approach to continuity which laid the groundwork for many significant breakthroughs in analysis. The modern treatment of continuity is typically attributed to Augustin-Louis Cauchy (1789-1857), but there are many more mathematicians who made important contributions in this direction as well. During this time integration theorists began to acquire tools which would allow them to rigorously define and study various integrals. In 1854 Bernhard Riemann (1826-1866) introduced the Riemann integral which was one of the first formally defined integrals. In 1904 Henri Lebesgue (1875-1941) was able to show that a real-valued function defined on a compact interval is Riemann integrable if and only if it is bounded and discontinuous on a set of Lebesgue measure 0. However, there are bounded derivatives which are discontinuous on a set of positive Lebesgue measure, and thus such derivatives are not Riemann integrable. An example of such a function was given by Vito Volterra (1860-1940) in 1881, which is constructed in (Lebesgue, 1902, Example 1.4.1). Drawback was remedied by the Lebesgue integral which Lebesgue introduced in

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(Lebesgue, 1902). As it turns out, the Lebesgue integral is a generalization of the Riemann integral and it is able to integrate all bounded derivatives restricted to compact intervals. Going back to our functions F and f where $F' = f$ on $[a,b] \setminus S$, if $[a,b]$ is compact then by Lebesgue's fundamental theorem of calculus we have that f is Lebesgue integrable on $[a,b]$ and F can be re-constructed by the Lebesgue integral of f if and only if F is absolutely continuous and S has Lebesgue measure 0. Note that the absolute continuity of F is a necessity here, thus it is easy to show that there are differentiable functions which cannot be recovered from their respective derivatives via the Lebesgue integral. If F is not absolutely continuous and S has Lebesgue measure 0, then in certain cases f may still be Lebesgue integrable but F can then not be recovered from the Lebesgue integral of f. For example, if $[a,b] = [0,1]$, F is the Cantor-Lebesgue function studied in (Lebesgue, 1904), and $f = 0$ every-where on $[0,1]$, then $F' = f$ on $[0,1] \setminus S$, where S has Lebesgue measure 0, yet F cannot be reconstructed from f via the Lebesgue integral. Unfortunately, the full power of the Lebesgue integral as it is known today was not understood by the mathematical community during this time and thus Lebesgue's work did not garner the immediate attention one perhaps would expect. On asecond thought this is not too surprising, since measure theory was still a very new topic and functional analysis had not yet been developed to a point where these two areas of mathematics could be researched in unity to produce a more complete theory. Perhaps this is a partial explanation to why Lebesgue was not appointed as professor until 1919, as mentioned in (Gowers, 2008). However, later on in the twentieth century mathematicians incrementally developed the necessary tools to study the Lebesgue integral more thoroughly and eventually the Lebesgue integral became the standard integral in advanced mathematical analysis. Despite the power of the Lebesgue integral, there were still finite and unbounded derivatives that could not be integrated. In 1912, Arnaud Denjoy (1884-1974) presented a powerful integral which was able to integrate all finite derivatives and recover their primitive functions. As described in (Liu et al., 2018, Section 1.1), loosely stated, Denjoy constructed a transfinite sequence $\{(I_k)\}_k \leq \Omega$ of increasingly general integrals, where Ω is the first uncountable ordinal, I_0 is the Lebesgue integral and I_Ω is the so called narrow Denjoy integral. Two years later in 1914 Oskar Perron (1880-1975) introduced an integral, which after some modification in 1915 by Hans Bauer also able to integrate all finite derivatives and recover their primitive functions. It took quite some time, but by 1925 mathematicians had realized that the narrow Denjoy integral and the Perron integral are in fact equivalent, this is called the Hake-Aleksandrov-Looman theorem. Consequently, the aforementioned integral of Denjoy and Perron is today called the Denjoy-Perron integral. Many years later in 1957, Jaroslav Kurzweil (born 1926) published a paper on differential equations in which he introduced a new integral. Four years later in 1961, while unaware of the work of Kurzweil, Ralph Henstock (1923-2007) published a paper on integration theory in which he introduced an integral which is primitive version of the integral of Jaroslav Kurzweil. Throughout a series of papers in the sixties, Henstock developed a substantial amount of properties of this integral. The definition of this integral as defined by Kurzweil in (Kurzweil, 1957) and Henstock in (Henstock, 1961) is quite elegant since it is highly reminiscent of the Riemann integral and since a substantial amount of its properties can be developed using Riemann sums and basic epsilon-delta proofs. Today the integral of Henstock and Kurzweil is called the Henstock-Kurzweil integral. As mathematicians later discovered, the Henstock-Kurzweil integral is in fact equivalent to the Denjoy-Perron integral. By the late nineties, a lot of integration theorists had researched the Henstock-Kurzweil integral extensively and consequently, the theory of this integral had been highly refined.

The Henstock–Kurzweil integral gives rise to a complete fundamental theorem of calculus in the sense that all finite derivatives can be integrated and their respective primitive functions can be re-constructed. This result is more general than the fundamental theorem of calculus for the Lebesgue integral which also requires the indefinite integral to be absolutely continuous. Henstock-Kurzweil integral gives rise to a monotone convergence theorem and a dominated convergence theorem, both of which are stronger than the corresponding theorems for the Lebesgue integral in the sense that weaker assumptions are imposed on the functional sequences in their respective suppositions. Improper Henstock-Kurzweil integral is contained in the Henstock-Kurzweil integral. This is yet another property that the Lebesgue integral does not possess. Thus, the theoretical deficiencies of the Lebesgue integral are by and large remedied by the Henstock-Kurzweil integral.

How we replace Henstock-Kurzweil (HK) integral with Lebesgue measure?

There are two ways to define an integral. One can provide a descriptive definition or an operational (or constructive). A descriptive definition describes the integral in relationship to its derivative without proving any process for its construction.

Definition 2.1 (Gill and Zachary, 2016): We define the weak variation $V(F,E)$, and the strong variation, $V_*\,(F,E)$, by:

$$V(FE) = \sup\left\{\sum_{i=1}^{n} |F(b_i) - F(a_i)|\right\}$$

and

$$V_*(F,E) = \sup\left\{\sum_{i=1}^{n} w(F,[a_i,b_i])\right\}$$

where the supremum is taken over all possible finite collections of non-overlapping intervals that have end points in E.

1. We say that F is of bounded variation on E, (BV), if $V(F,E) < \infty$.
2. We say that F is of restricted bounded variation on E, (BV_*), if $V_*(F,E) < \infty$.
3. We say that F is absolutely continuous on E, (AC), if for each $\epsilon > 0$, there exists a $\delta > 0$ such that, for every collection $\{[a_i\,, b_i], 1 \le i \le n\}$, of nonoverlapping intervals with end points in E and $\sum_{i=1}^{n}(b_i - a_i) < \delta$, then $\sum_{i=1}^{n}|F(b_i) - F(a_i)| < \epsilon$.
4. We say that F is absolutely continuous on E in the restricted sense, $(AC)_*$, if for each $\epsilon > 0$, there exists a $\delta > 0$ such that, for every collection $\{[a_i, b_i], 1 \le i \le n\}$, of nonoverlapping intervals with endpoints in E and $\sum_{i=1}^{n}(b_i - a_i) < \delta$, then $\sum_{i=1}^{n} w(F,[a_i,b_i]) < \epsilon$.
5. We say that F is generalized absolutely continuous on E, (ACG), if $F|_E$ is continuous and E is a countable union of sets $\{E_i\}$ such that F is (AC) on each E_i.
6. We say that F is generalized absolutely continuous in the restricted sense in E, $(ACG)_*$, if $F|_E$ is continuous and E is a countable union of sets $\{E_i\}$ such that F is $(AC)_*$ on each E_i.

The one thing that the Riemann approach in a first year analysis course cannot do is allow an immediate entry into measure theory in the subsequent course. However, even here things are more interesting than this simple remark would suggest. Due to an idea of Henstock, developed in particular by Thomson, (Thomson 1981), there are associated with each Riemann approach a series of natural metric measures that on analysis are found to contain the basic information about the primitives more precisely than the classical concepts of ACG_* etc. In particular the concept of variation allows yet another way of defining the various Riemann integrals. These ideas are a little too refined for full mention here, but are part of the excellent re-working of trigonometric integrals by Thomson (Thomson 1981).

Definition 2.2 Descriptive Definitions: Let E be a measurable subset of $\mathbb{R}$ and $\mu(E)$ denote the Lebesgue measure of E. Let $c \in \mathbb{R}$.

1. We say that c is a point of density for E if
$$d_cE = \lim_{h\to 0^+} \frac{\mu\big(E \cap (c-h, c+h)\big)}{2h} = 1.$$
2. We say that c is a point of dispersion for E if
$$d_cE = \lim_{h\to 0^+} \frac{\mu(E \cap (c-h, c+h))}{2h} = 0.$$
3. We say that a function $F{:}\,[a,b] \to \mathbb{R}$ is approximately continuous at $c \in E \subset [a,b]$, if c is a point of density for E and $F|_E$ is differentiable at c.

Theorem 2.3 (Gill and Zachary, 2016): Let F be a function defined on $[a,b]$ with $F(a)=0$, then the following holds:

1. If F is (AC) on $[a,b]$, then F' exists $a.e.$, and F' is Lebesgue integrable , then
$$\int_a^x F'(y)d\mu(y) = F(x).$$
2. If F is (ACG_*) on $[a,b]$, then F' exists $a.e.$, and F' is Henstock-Kurzweil integrable, then
$$\int_a^x F'(y)d\mu(y) = F(x).$$

Example 2.4: If F is any interval function and if $V_*F(E) = \inf_{\delta}\sup_{P}\sum|F(J_i)|$ where δ is a gauge and P is a partition in I that is anchored in the set E, then V_* is a regular Borel measure.
Example 2.5: If V_* is AC then F is the HK-integral of its derivative, that exists almost everywhere; further, if in addition V_* is finite then F is the Lebesgue integral of this derivative.

Application of Henstock-Kurzweil integral and non absolute integrable function spaces

The HK integral is one of the most powerful methods of integration currently being researched by the mathematicians. By using gauges, one can evaluate functions on more of a local level than one can with the traditional Riemann integral. This seemingly small change to the traditional definition of the Riemann integral has proven to have far reaching consequences. For example, the HK integral makes integration and differentiation truly inverse processes. The fact that the HK integral is a non-absolutely convergent integral makes it ideal for integrating functions which oscillate wildly, a feature not always available with the Lebesgue integral.

This allows one to look at the integration process as a whole, rather than being forced to consider the negative and nonnegative cases separately as is often the case in Lebesgue integration theory. However, this advantage does have its drawbacks. To date no one has developed a suitable norm for the space $HK(I)$.

Despite the power of the Henstock-Kurzweil integral it is rarely used in under graduate or even postgraduate courses on mathematical analysis. There are several reasons for this. In elementary analysis courses students typically study the

Darboux integral (and frequently take the Cauchy criterion for Darboux integrability as the definition of the integral). It is well-known that the Darboux integral is equivalent to the Riemann integral. First year students have a tendency of thinking in terms of formulas and graph representations of rather well-behaved functions and pay little to no attention to further details. Thus, a somewhat formal treatment of the Henstock-Kurzweil integral does certainly not belong in an analysis textbook aimed at first year students. In fact, Peng Yee Lee (born 1938) who was a student under Henstock between 1961 and 1965 mentioned in (Henstock, 1961) that Henstock once made an attempt to teach the Henstock-Kurzweil integral to first year students. Apparently, this was a disaster and he never tried it again. Since the Henstock-Kurzweil integral is in some sense a natural extension of the Riemann integral, one could argue that perhaps it should be introduced at a later stage. However, in postgraduate analysis courses (and perhaps in some cases in advanced undergraduate analysis courses) the integral of choice is for the most part the Lebesgue integral. While a substantial amount of the theory of the Lebesgue integral can be developed from the Henstock–Kurzweil integral, it is likely that the Henstock–Kurzweil integral has to contribute something highly practical which the Lebesgue integral does not, in order for it to have a chance of at least partially replacing the Lebesgue integral in advanced analysis courses. Some research involving the Henstock-Kurzweil integral, partial differential equations and integral transformations has been done, see for example (Mema, 2013; Mohanty and Talvila, 2003). Sometimes we are interested in solving problems with minimal smoothness assumptions and the solutions to such problems might involve bad functions that the theory of Lebesgue cannot deal with. Perhaps the Henstock-Kurzweil integral

could come into play here and remedy some of these problems. However, it will take more than three references to seemingly arbitrary papers in order for this to be the case. There needs to be a substantial amount of theory developed which is accessible to postgraduate students and which brings something new to the field of differential equations and integral transformations in order for the Henstock–Kurzweil integral to be seen more frequently in postgraduate courses. Whether or not this is possible is a question which is far beyond the scope of this thesis. Given what we have discovered in this thesis, even though a lot of results are quite interesting, it is difficult to argue that the Henstock–Kurzweil integral should be taught more in advanced analysis courses.
In several physical phenomena, highly oscillating or singular functions appear (Hamed and Cummins, 1991; Condon et al. 2009; Hong and Xu, 2001). The Lebesgue integral is not enough for some highly oscillating functions leading to the possibility that the integral on the right side of the equality 2 (Becerra et al., 2020) does not exist for this type of functions and so the variational problem 2 (Becerra et al., 2020) would not be well defined. One way to solve this problem is to change the type of integral to be considered, in this work we will use the Henstock-Kurzweil integral. One way to solve this problem is to change the type of integral to be considered, in this work we will use the Henstock-Kurzweil integral. Different authors have studied differential equations involving Henstock-Kurzweil integrable functions. In (Len-Velasco et al., 2019) the authors use the Finite Element Method (FEM) for finding numerical solutions of elliptic problems with Henstock-Kurzweil integrable functions. They use open quadrature and Lobatto quadrature to approximate numerically the integrals that appear in the FEM. In (Liu et al., 2018) are given conditions to establish the existence of a solution to nonlinear second-order differential equations of type:

$$-D^2x = f(t,x) + g(t,x)Du$$

subject to the boundary conditions $x(0) = \beta\, Dx(0), Dx(1) + Dx(v) = 0$, where the derivatives are in the distributional sense, x, u are regulated functions and g is of bounded variation. In (Becerra et al., 2020), the Henstock-Kurzweil-Stieltjes integral is used to transform the distributional differential equation into an integral equation, then the Leray-Schauder nonlinear alternative theorem is applied for finding a solution.

In (Sanchez-Perales and Mendoza-Torres, 2020) the existence and uniqueness of the Shrödinger equation, $-y'' + qy = f$ a.e. on $[a,b]$ subject to arbitrary boundary values, is guaranteed for functions f, q Henstock-Kurzweil integrable. Properties of the inverse of the Shrödinger operator are established, then the authors give conditions so that the solution of the differential equation can be expressed as a Fourier type series.
Gill and Zachary (Gill and Zachary, 2016) discussed the space $KS^p(\mathbb{R}^n)$ called Kuelbs-Steadman space. In Feynman path integral: The properties of $\mathbb{KS}^p[\boldsymbol{R}^n]$ derived earlier suggests that $\mathbb{KS}^2[\boldsymbol{R}^n]$ may be a better Hilbert space than $\boldsymbol{L}[\boldsymbol{R}^n]$ for the study of the path integral formulation of quantum theory developed by Feynman. Note that it is easy to prove that both the position and momentum operators have closed densely defined extensions to $\mathbb{KS}^2[\boldsymbol{R}^n]$. Furthermore, the extensions of Fourier, $\mathfrak{F}$ and Convolution, $\mathfrak{C}$ insure that all of the Schrödinger and Heisenberg theory has a faithful representation on $\mathbb{KS}^2[\boldsymbol{R}^n]$. These issues will be discussed more fully in another venue.
In operator theory on separable Hilbert spaces, the major problem with integration for operator valued functions is that these functions need not have a Lebesgue (like) integral. However, they always have a HK-integral (Hille and Phillips, 1957).

References

Hamed, M.A., Cummins, B., 1991. A numerical integration formula for the solution of the singular integral equation for classical crack problems in plane and antiplane elasticity. *Journal of King Saud University - Engineering Sciences* 3, 217–230.

Becerra, T.P., Sánchez-Perales, S., Oliveros, J.J. 2020. The HK-Sobolev space and applications to one-dimensional boundary value problems, *Journal of King Saud University*, https://doi.org/10.1016/j.jksus.2020.06.016

Grabiner, J.V. 1983. Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus. *The American Mathematical Monthly*, (3), 185–194

Thomson, Brian S., Bruckner, Judith B, Buckner, M. Andrew M. 2008. Elementary Real Analysis. Prentice Hall, Second Edition.

Condon, M., Deaño, A., Iserles, A., 2009. On highly oscillatory problems arising in electronic engineering. M2AN Math. Model. Numer. Anal. 43, 785–804.

Henri Lebesgue. 1902. Intégrale, longueur, aire. Annali di Matematica Pura ed Ap-plicata, vol. 7 (1), 231–359.

Henri Lebesgue. 1904. Leçons sur l'intégration et la recherche des fonctions primi-tives. Gauthier-Villars, Paris.

Liu, W., Ye, G., Zhao, D., Torres, D.F.M., 2018. Existence theorems for a nonlinear second-order distributional differential equation. Journal of King Saud University-Science 30, 527–530.

Hong, J., Xu, A. 2001. Effects of gravity and nonlinearity on the waves in the granular chain. Phys Rev E Stat Nonlin Soft Matter Phys 63, 061310.

Oleksiy Dovgoshey, Olli Martio, Vladimir I. 2006. Ryazanov and Matti Vuorinen. The Cantor Function. Expositiones Mathematicae, 24(1), 1–37.

Gill. T.L., Zachary, W.W., 2016. Functional Analysis and the Feynman operator Calculus, Springer, New York.

Gower. Timothy., Leader. Imre , 2008. The Princeton Companion to Mathematics. Princeton University Press, New Jersey.

Hoffmann, Heiko., 2014. Descriptive Characterisation of the Variational Henstock–Kurzweil–Stieltjes Integral and Applications. Karlsruhe Institute of Technology, Faculty of Mathematics, Institute for Analysis..

E. Hille, E.,Phillips, R.S., 1957. Functional Analysis and Semigroups. American Mathematical Society Colloquium Publications, vol. 31 American Mathematical Society, Providence, RI.

Kurzweil, Jaroslav., 1957. Generalized Ordinary Differential Equations and Continuous Dependence on a Parameter. Czechoslovak Mathematical Journal, 7(3) 418–449.

Henstock. R, 1961. Definitions of Riemann Type of the Variational Integrals. Proceedings of the London Mathematical Society, 11(1) 402–418.

Sánchez-Perales, S., Mendoza-Torres, F.J., 2020. Boundary value problems for the Schrödinger equation involving the Henstock-Kurzweil integral. Czechoslovak Math. J. 70, 519–537.

Mema. Esterina, 2013. Equiintegrability and Controlled Convergence for the Henstock–Kurzweil Integral. International Mathematical Forum, 8(19), 913–919.

Mohanty. Parasar, Talvila, Erik, 2004. A Product Convergence Theorem for Henstock–Kurzweil Integrals. Real Analysis Exchange, 29(1), 199–204.

Peng. Yee Lee, 1989-90. On ACG Functions. Real Analysis Exchange, 15(2), 754–759.

Gordon. Russell A, 1994. The Integrals of Lebesgue, Denjoy, Perron and Henstock. American Mathematical Society, Providence, RI.

Salvador. Sánchez–Perales, and Tenorio. Jesus. F., 2014. Laplace Transform Using the Henstock-Kurzweil Integral. Revista de la Unión Matemática Argentina, 55(1) 7-15

León-Velasco, D.A., Morín-Castillo, M.M., Oliveros-Oliveros, J.J., Pérez-Becerra, T., Escamilla-Reyna, J.A., 2019. Numerical solution of some differential equations with Henstock-Kurzweil functions. Journal of Function Spaces, Art. ID 89485709.