Dera Natung Government College Research Journal
Volume 6 Issue 1, pp. 92-102

ISSN: 2456-8228
January-December 2021

Research article

On Λ-Fibonacci difference sequence spaces of fractional order

Taja Yaying*
Dera Natung Government College, Itanagar-791113, Arunachal Pradesh, India.

***Abstract*.** In this article, we introduce Λ-Fibonacci difference operator of fractional order $\hat{\mathcal{F}}_\lambda(\Delta^{(\alpha)})$ which is obtained by the composition of Λ -Fibonacci matrix $\hat{\mathcal{F}}_\lambda$ and backward fractional difference operator $\Delta^{(\alpha)}$, defined by

$$\left(\Delta^{(\alpha)}x\right)_k = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha-i+1)} x_{k-i},$$

and introduce the sequence spaces $c_0^\lambda(\hat{\mathcal{F}}(\Delta^{(\alpha)}))$ and $c^\lambda(\hat{\mathcal{F}}(\Delta^{(\alpha)}))$. We give some topological properties, and obtain the Schauder basis of the new spaces.

***Keywords*.** Fibonacci sequence, Λ-sequence, Difference operator $\Delta^{(\alpha)}$, Λ-Fibonacci difference sequence spaces, Schauder basis.
***AMS subject classification (2010)*.** 46A45, 46B45.

1 Introduction

The number sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ is known as the Fibonacci sequence. Note that any number in the sequence is the sum of the two numbers preceding it. Thus, if $(f_n)_{n=0}^{\infty}$ is the sequence of Fibonacci numbers, then,

$$f_0 = f_1 = 1, \text{and } f_n = f_{n-1} + f_{n-2}, \qquad n \geq 2.$$

The ratio of the successive terms in the Fibonacci sequence approaches to an irrational number $\frac{1+\sqrt{5}}{2}$, called the golden ratio. This number has a great application in the field of architecture, science and arts. Some more basic properties of Fibonacci numbers (Koshy, 2001) can be listed as follows:

$$\lim_{n\to\infty}\frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} \ \ (golden\ ratio),$$

$$\sum_{k=0}^{n} f_k = f_{n+2} - 1 \ \ (n \in \mathbb{N}_0),$$

**Corresponding Author: tajayaying20@gmail.com*
Received on: 17.10.2021, Accepted on: 18.02.2022
Cite as: ***Yaying, T. 2021. On Λ-Fibonacci difference sequence spaces of fractional order, Dera Natung Government College Research Journal, 6, 92-102.***
DOI: https://doi.org/10.56405/dngcrj.2021.06.01.10

$$\sum_k \frac{1}{f_k} \quad converges,$$

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1}, n \geq 1 \ (Cassini\ formula).$$

Here and throughout $\mathbb{N} = \{0,1,2,3, \dots\}$ and ω is the space of all real valued sequences. By ℓ_∞, c_0 and c, we mean the spaces all bounded, null and convergent sequences respectively, normed by $\|x\|_\infty = \sup_k |x_k|$. Also by ℓ_1, cs, and bs, we mean the spaces of absolutely summable, convergent series and bounded series, respectively, where $1 \leq p < \infty$. A Banach space X is said to be a BK space if each projection $x \mapsto x_n$ on the n^{th} coordinate is continuous. A BK space $X \supseteq \emptyset$ is said to have AK if $\sum_{k=0}^{m} x^k e^{(k)} \to x \ (m \to \infty)$ for every sequence $x = (x_k) \in X$, where $e = (1,1,1, \dots)$ and $e^{(k)}$ is the sequence whose only non-zero term is 1 in the k^{th} place for each $k \in \mathbb{N}_0$. Here and henceforth, for simplicity in notation, the summation without limit runs from 0 to ∞.

1.1 Literature Review

The notion of difference sequence space was introduced by Kızmaz (Kızmaz, 1981). Later on, the notion was generalized by Et and Çolak (Et and Çolak, 1995) as given below:

Let m be a non negative integer, then

$$\Delta^m(X) = \{x = (x_k) : \Delta^m x \in X \text{ for } X \in \{\ell_\infty, c, c_0\}\},$$

where $(\Delta^m x)_k = (\Delta^{m-1}x)_k - (\Delta^{m-1}x)_{k+1}$, $\Delta^0 x = x$, and

$$(\Delta^m x)_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}.$$

These spaces are Banach spaces with norm defined by

$$\|x\|_{\Delta^m} = \sum_{i=0}^{m} |x_i| + \sup_k |(\Delta^m x)_k|.$$

Furthermore, generalized difference sequence space was studied by Et and Esi (Et and Esi, 2000), Et and Basarır (Et and Basarır, 1997), Malkowsky and Parashar (Malkowsky and Parashar, 1997), Bektas et al. (Bektas et al., 2004) and many others.

Let X and Y be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex entries. We write A_n and A^k to denote the sequences in the n^{th} row and k^{th} column of the matrix A, respectively. We recall that A defines a matrix mapping from X to Y if for every sequence $x = (x_k)$, the A- transform of x i.e. $Ax = \{(Ax)_n\}_{n=0}^{\infty} = \{\sum_k a_{nk}x_k\}_{n=0}^{\infty} \in Y$.

The sequence space X_A defined by

$$X_A = \{ x = (x_k) \in \omega : Ax \in X\}, \tag{1}$$

is called the domain of matrix A.

By (X, Y), we denote the class of all matrices A from X to Y. Thus $A \in (X, Y)$ if and only if for each $x \in X$ such that $Ax \in Y$.

Fibonacci double band matrix $\hat{\mathcal{F}} = (\hat{f}_{nk})$ is defined by \begin{equation*}

$$\hat{f}_{nk} = \begin{cases} \frac{f_{n+1}}{f_n}, & k = n-1, \\ \frac{f_n}{f_{n+1}}, & k = n, \\ 0, & otherwise. \end{cases}$$

Kara (Kara, 2013) defined the sequence space $\ell_p\,(\hat{\mathcal{F}})$, $1 \leq p \leq \infty$ as follows:

$$\ell_p\left(\hat{\mathcal{F}}\right) = \{x = (x_n) \in \omega : \sum_n \left|\frac{f_n}{f_{n+1}}x_n - \frac{f_{n+1}}{f_n}x_{n-1}\right|^p < \infty\}, (1 \leq p < \infty),$$

$$\ell_\infty\left(\hat{\mathcal{F}}\right) = \{x = (x_n) \in \omega : \sup_{n\in\mathbb{N}} \left|\frac{f_n}{f_{n+1}}x_n - \frac{f_{n+1}}{f_n}x_{n-1}\right| < \infty\}.$$

Later on, Başarır et al. (Başarır et al., 2016) studied the Fibonacci difference sequence spaces $c_0(\hat{\mathcal{F}})$ and $c(\hat{\mathcal{F}})$\$ as the set of all sequences whose $\hat{\mathcal{F}}$-transform are in the spaces c_0 and c, respectively. Since then many authors studied and generalized the Fibonacci difference sequence spaces, for instance, Candan (Candan, 2015), Kara and Demiriz (Kara and Demiriz, 2015), Das and Hazarika (Das and Hazarika, 2017), and Alotaibi et al. (Alotaibi et al., 2015).

By $\Gamma(m)$, we denote the Gamma function of all real numbers $m \notin \{0, -1, -2, \ldots\}$. $ $\Gamma(m)$ can be defined via a convergent improper integral given by

$$\Gamma(m) = \int_0^\infty e^{-x} x^{m-1} dx. \tag{2}$$

By using equality (2), we state some basic properties of Gamma function which are used throughout the text:

1. For $m \in \mathbb{N}$, $\quad \Gamma(m+1) = m!$.
2. For any real number $m \notin \{0, -1, -2, \ldots\}$, $\quad \Gamma(m+1) = m\Gamma(m)$.
3. For particular cases, we have $\Gamma(1) = \Gamma(2) = 1, \Gamma(3) = 2!, \Gamma(4) = 3!, \ldots$.

For a positive proper fraction α, Baliarsingh (Baliarsingh, 2013) (see also (Baliarsingh and Dutta, 2015a, 2015b)) defined a generalized fractional difference operator $\Delta^{(\alpha)}$ and its inverse by

$$\left(\Delta^{(\alpha)} x\right)_k = \sum_i (-1)^i \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha-i+1)} x_{k-i}, \tag{3}$$

$$(\Delta^{-\alpha} x)_k = \sum_i (-1)^i \frac{\Gamma(-\alpha+1)}{i!\,\Gamma(-\alpha-i+1)} x_{k-i}. \tag{4}$$

Throughout, it is assumed that the series on the right hand side of equalities (3) and (4) are convergent for $x = (x_k) \in \omega$. It is more convenient to express $\Delta^{(\alpha)}$ as a triangle

$$\Delta_{nk}^{(\alpha)} = \begin{cases} \sum_i (-1)^{n-k} \dfrac{\Gamma(\alpha+1)}{(n-k)!\,\Gamma(\alpha-n+k+1)}, & (0 \le k \le n), \\ 0, & (k > n), \end{cases}$$

For more study on fractional difference operator, one may refer to (Yaying, 2019, 2012), (Yaying et al., 2021a, 2021b).

2. Λ-Fibonacci difference sequence spaces of fractional order

Throughout we assume that $\lambda = (\lambda_k)$ is a strictly increasing sequence of positive reals tending to ∞, that is,

$$0 < \lambda_0 < \lambda_1 < \cdots \text{ and } \lim_{k\to\infty} \lambda_k = \infty.$$

Mursaleen and Noman (Mursaleen and Noman, 2010a) defined the infinite matrix $\Lambda = (\lambda_{nk})$ by

$$\lambda_{nk} = \begin{cases} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n}, & (0 \leq k \leq n), \\ 0, & (k > n). \end{cases}$$

By using the Λ-matrix, Mursaleen and Noman (Mursaleen and Noman, 2010a) introduced the Λ-sequence spaces c_0^λ and c^λ as follows:

$$c_0^\lambda = \{x = (x_k) \in \omega : (\Lambda x)_n \in c_0 \};$$

$$c^\lambda = \{x = (x_k) \in \omega : (\Lambda x)_n \in c \}.$$

Later on, Mursaleen and Noman (Mursaleen and Noman, 2010b) studied Λ-difference sequence spaces $c_0^\lambda(\Delta)$ and $c^\lambda(\Delta)$.

Recently, by using the Λ matrix and the Fibonacci matrix $\hat{\mathcal{F}}$, Das and Hazarika (Das and Hazarika, 2017) defined the product matrix $\mathcal{F}_\lambda = \Lambda\hat{\mathcal{F}} = (\bar{f}_{nk})$ by

$$\bar{f}_{nk} = \begin{cases} \dfrac{1}{\lambda_n}\left\{(\lambda_k - \lambda_{k-1})\dfrac{f_k}{f_{k+1}} - (\lambda_{k+1} - \lambda_k)\dfrac{f_{k+2}}{f_{k+1}}\right\}, & 0 \leq k < n, \\ \dfrac{1}{\lambda_n}(\lambda_n - \lambda_{n-1})\dfrac{f_n}{f_{n+1}}, & k = n, \\ 0, & otherwise, \end{cases}$$

and studied the sequence spaces $c_0^\lambda(\hat{\mathcal{F}})$ and $c^\lambda(\hat{\mathcal{F}})$. Later on, Das et al. (Das et al., 2022) also studied the sequence spaces $\ell_p^\lambda(\hat{\mathcal{F}})$ and $\ell_\infty^\lambda(\hat{\mathcal{F}})$. We extend the notion of Das and Hazarika (Das and Hazarika, 2017) and introduce Λ-Fibonacci difference sequence spaces of fractional order.

Now, combining the matrix $\mathcal{F}_\lambda = \Lambda\hat{\mathcal{F}}$ and difference operator $\Delta^{(\alpha)}$, we obtain Λ-Fibonacci difference operator of fractional order α, denoted by $\mathcal{F}_\lambda\Delta^{(\alpha)} = (\tilde{f}_{nk})$ and defined by

$$\tilde{f}_{nk} = \begin{cases} \displaystyle\sum_{j=k}^{n-1}(-1)^{j-k}\frac{\Gamma(\alpha+1)}{(j-k)!\,\Gamma(\alpha-j+k+1)}\frac{1}{\lambda_n}\left\{\frac{f_j(\lambda_j - \lambda_{j-1})}{f_{j+1}} - \frac{f_{j+1}(\lambda_{j+1} - \lambda_j)}{f_{j+2}}\right\} + (-1)^{n-k}\frac{\Gamma(\alpha+1)}{(n-k)!\,\Gamma(\alpha-n+k+1)}\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\frac{f_n}{f_{n+1}}, & 0 \leq k < n, \\ \dfrac{1}{\lambda_n}(\lambda_n - \lambda_{n-1})\dfrac{f_n}{f_{n+1}}, & k = n, \\ 0, & otherwise, \end{cases}$$

Now we give certain results regarding the inverses of the matrices $\Delta^{(\alpha)}$, $\mathcal{F}_\lambda$ and $\mathcal{F}_\lambda\Delta^{(\alpha)}$):

Lemma 2.1 (Baliarsingh and Dutta, 2015a): The inverse of the difference matrix $\Delta^{(\alpha)}$ is given by the triangle

$$\Delta_{nk}^{(-\alpha)} = \begin{cases} (-1)^{n-k} \dfrac{\Gamma(-\alpha+1)}{(n-k)!\,\Gamma(-\alpha-n+k+1)}, & (0 \le k \le n), \\ 0, & (k > n). \end{cases}$$

Lemma 2.2 (Das and Hazarika, 2017): The inverse of the Λ-Fibonacci matrix $\mathcal{F}_\lambda^{-1} = (\bar{f}_{nk}^{-1}\})$ is given by the triangle

$$\bar{f}_{nk}^{-1} = \begin{cases} \lambda_k f_{n+1}^2 \left\{ \dfrac{1}{\lambda_k - \lambda_{k-1}} \dfrac{1}{f_k f_{k+1}} - \dfrac{1}{\lambda_{k+1} - \lambda_k} \dfrac{1}{f_{k+1} f_{k+2}} \right\}, & 0 \le k < n, \\ \dfrac{\lambda_n}{\lambda_n - \lambda_{n-1}} \dfrac{1}{f_n}, & k = n, \\ 0, & otherwise, \end{cases}$$

Lemma 2.3:

The inverse of the product matrix $\mathcal{F}_\lambda \Delta^{(\alpha)}$ is given by

$$\tilde{f}_{nk} = \begin{cases} (-1)^{n-k} \dfrac{\Gamma(-\alpha+1)}{(n-k)!\,\Gamma(-\alpha-n+k+1)} \dfrac{\lambda_k f_{k+1}}{(\lambda_k - \lambda_{k-1}) f_k} + l(f,\lambda;n), & 0 \le k < n, \\ \dfrac{\lambda_n}{(\lambda_n - \lambda_{n-1})} \dfrac{f_{n+1}}{f_n}, & k = n, \\ 0, & otherwise, \end{cases}$$

where

$$l(f,\lambda;n) = \sum_{j=k+1}^{n} (-1)^{n-j} \frac{\Gamma(-\alpha+1)}{(n-j)!\,\Gamma(-\alpha-n+j+1)} \left\{ \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} \frac{f_{j+1}^2}{f_k f_{k+1}} - \frac{\lambda_k}{\lambda_{k+1} - \lambda_k} \frac{f_{j+1}^2}{f_{k+1} f_{k+2}} \right\}.$$

Proof: The result can be obtained by using Lemmas 2.1 and 2.2.

By using the matrix $\mathcal{F}_\lambda \Delta^{(\alpha)}$, we define the sequence spaces $c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ and $c^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ as the set of all sequences whose $\mathcal{F}_\lambda \Delta^{(\alpha)}$-transforms are in the spaces c_0 and c, respectively. That is

$$c_0^\lambda\big(\hat{\mathcal{F}}\Delta^{(\alpha)}\big) = \{\, x = (x_n) \in \omega\colon\ \mathcal{F}_\lambda \Delta^{(\alpha)} x \in c_0 \},$$

$$c^\lambda\big(\hat{\mathcal{F}}\Delta^{(\alpha)}\big) = \{\, x = (x_n) \in \omega\colon\ \mathcal{F}_\lambda \Delta^{(\alpha)} x \in c \}.$$

Thus, we can write

$$c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)}) = (c_0)_{\mathcal{F}_\lambda\Delta^{(\alpha)}} \; and \; c^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)}) = c_{\mathcal{F}_\lambda\Delta^{(\alpha)}}. \tag{5}$$

It is clear that the sequence spaces $c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ and $c^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ may be reduced to certain classes of sequence spaces in the special cases of $\alpha \in \mathbb{R}$.

1. For $\alpha = 0$, the above sequence spaces reduce to the classes $c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ and $c^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ as defined by Das and Hazarika (Das and Hazarika, 2017).
2. For $\alpha = 1$, the above sequence spaces reduce to $c_0^\lambda(\hat{\mathcal{F}}\Delta)$ and $c^\lambda(\hat{\mathcal{F}}\Delta)$.
3. For $\alpha = m \in \mathbb{N}$, the above sequence spaces reduce to the classes $c_0^\lambda(\hat{\mathcal{F}}\Delta^{(m)})$ and $c^\lambda\left(\hat{\mathcal{F}}\Delta^{(m)}\right)$.

Now, we define the sequence $y = (y_k)$ which will be frequently used as the $\mathcal{F}_\lambda\Delta^{(\alpha)}$-transform of the sequence $x = (x_k)$

$$y_k = \sum_{j=0}^{k-1}\left[\sum_{i=j}^{k-1}(-1)^{i-j}\frac{\Gamma(\alpha+1)}{(i-j)!\,\Gamma(\alpha-i+j+1)}\left\{\frac{\lambda_i-\lambda_{i-1}}{\lambda_k}\frac{f_i}{f_{i+1}}-\frac{\lambda_{i+1}-\lambda_i}{\lambda_k}\frac{f_{i+2}}{f_{i+1}}\right\}\right. \tag{6}$$
$$\left. + (-1)^{k-j}\frac{\Gamma(\alpha+1)}{(k-j)!\,\Gamma(\alpha-k+j+1)}\frac{\lambda_k-\lambda_{k-1}}{\lambda_k}\frac{f_k}{f_{k+1}}\right]x_j + \frac{\lambda_k-\lambda_{k-1}}{\lambda_k}\frac{f_k}{f_{k+1}}x_k$$

for each $k \in \mathbb{N}_0$. We begin with the following theorem:

Theorem 2.4: The sequence spaces $c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ and $c^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ are BK-spaces with the norm defined by

$$\|x\|_{c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})} = \|x\|_{c^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})} = \left\|\mathcal{F}_\lambda\Delta^{(\alpha)}x\right\|_{\ell_\infty} = \sup_{n\in\mathbb{N}_0}\left|\left(\mathcal{F}_\lambda\Delta^{(\alpha)}x\right)_n\right|.$$

Proof: The sequence spaces c_0 and c are BK spaces with their natural norms. Since equality (5) holds and $\mathcal{F}_\lambda\Delta^{(\alpha)}$ is a triangular matrix, therefore Theorem 4.3.12 of Wilansky (Wilansky, 1984) yields the fact that $c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ and $c^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ are BK-spaces with respect to the given norm.

Theorem 2.5: The sequence spaces $c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ and $c^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ are linearly isomorphic to c_0 and c, respectively.

Proof: We prove the result for the space $c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$. Define a mapping $T: c_0^\lambda\left(\hat{\mathcal{F}}\Delta^{(\alpha)}\right) \to c_0$ by $x \mapsto y = Tx = \mathcal{F}_\lambda\Delta^{(\alpha)}x$. Clearly, T is linear and $x = 0$ whenever $Tx = 0$. Thus, T is injective.

Let $y = (y_k) \in c_0$ and by using Lemma 2.3, we define a sequence $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k-1} \left[(-1)^{k-j} \frac{\Gamma(-\alpha+1)}{(k-j)!\,\Gamma(-\alpha-k+j+1)} \frac{\lambda_j f_{j+1}}{(\lambda_j - \lambda_{j-1}) f_j} + \sum_{i=j+1}^{k} (-1)^{k-i} \frac{\Gamma(-\alpha+1)}{(k-i)!\,\Gamma(-\alpha-k+i+1)} \left\{ \frac{\lambda_j}{\lambda_j - \lambda_{j-1}} \frac{f_{i+1}^2}{f_j f_{j+1}} - \frac{\lambda_j}{\lambda_{j+1} - \lambda_j} \frac{f_{i+1}^2}{f_{j+1} f_{j+2}} \right\} \right] y_j + \frac{\lambda_k f_{k+1}}{(\lambda_k - \lambda_{k-1}) f_k} y_k \tag{7}$$

for each $k \in \mathbb{N}_0$. Then, we have

$$\begin{aligned} &\lim_{k\to\infty} \left(\mathcal{F}_\lambda \Delta^{(\alpha)} x \right)_k \\ &= \lim_{k\to\infty} \left(\sum_{j=0}^{k-1} \left[\sum_{i=j}^{k-1} (-1)^{i-j} \frac{\Gamma(\alpha+1)}{(i-j)!\,\Gamma(\alpha-i+j+1)} \left\{ \frac{\lambda_i - \lambda_{i-1}}{\lambda_k} \frac{f_i}{f_{i+1}} - \frac{\lambda_{i+1} - \lambda_i}{\lambda_k} \frac{f_{i+2}}{f_{i+1}} \right\} + (-1)^{k-j} \frac{\Gamma(\alpha+1)}{(k-j)!\,\Gamma(\alpha-k+j+1)} \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} \frac{f_k}{f_{k+1}} \right] x_j + \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} \frac{f_k}{f_{k+1}} x_k \right) \\ &= \lim_{k\to\infty} y_k = 0. \end{aligned}$$

Therefore, $x \in c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ and $y = Tx$. Consequently, T is surjective and also norm preserving. Thus $c_0^\lambda\left(\hat{\mathcal{F}}\Delta^{(\alpha)}\right) \cong c_0$.

3. Schauder Basis

In this section, we shall construct Schauder basis for the sequence spaces $c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ and $c^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$.

A sequence $x = (x_k)$ of a normed space $(X, \|\cdot\|)$ is called a Schauder basis if for every $u \in X$ there exist a unique sequence of scalars $a = (a_k)$ such that

$$\lim_{n\to\infty} \left| u - \sum_{k=0}^{n} a_k x_k \right| = 0.$$

It is known that the set $\{e^{(k)}\}$ is the Schauder basis for the space c_0. The mapping $T: c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)}) \to c_0$ defined in the proof of Theorem 2.5 is an isomorphism, therefore the inverse image of the set $\{e^{(k)}\}$ forms the basis for the new space $c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$. We give the following results:

Theorem 3.1: Let $\xi_k(\lambda) = \left(\mathcal{F}_\lambda\Delta^{(\alpha)}x\right)_k$ for all $k \in \mathbb{N}_0$. Define the sequence $b^{(k)}(\lambda) = \{b_n^{(k)}(\lambda)\}$ of the elements $c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ for every fixed $k \in \mathbb{N}_0$ by

$$b_n^{(k)}(\lambda) = \begin{cases} (-1)^{n-k}\dfrac{\Gamma(-\alpha+1)}{(n-k)!\,\Gamma(-\alpha-n+k+1)}\dfrac{\lambda_k f_{k+1}}{(\lambda_k-\lambda_{k-1})f_k} + l(f,\lambda;n), & k < n, \\ \dfrac{\lambda_n}{(\lambda_n-\lambda_{n-1})}\dfrac{f_{n+1}}{f_n}, & k = n, \\ 0, & otherwise, \end{cases}$$

Then

(a) the sequence $b^{(k)}(\lambda)$ is a Schauder basis for the sequence space $c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ and every $x \in c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ has a unique representation of the form

$$x = \sum_k \xi_k(\lambda) b_n^{(k)}(\lambda).$$

(b) the set $\left\{\left(\mathcal{F}_\lambda\Delta^{(\alpha)}\right)^{-1}e, b^{(k)}(\lambda)\right\}$ is a Schauder basis for the space $c^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ and every $x \in c^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ has a unique representation of the form

$$x = le + \sum_k |\xi_k(\lambda) - l|,$$

where $l = \lim_{k\to\infty} \xi_k(\lambda)$ for each $k \in \mathbb{N}_0$.

Corollary 3.2: The sequence spaces $c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ and $c^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ are seperable.

Proof: The result is immediate from Theorems 2.4 and 3.1.

4. Conclusion

This is article is devoted to construction of Λ-Fibonacci difference sequence spaces $c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ and $c^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ of fractional order, by using the product of Λ-Fibonacci matrix and difference operator $\Delta^{(\alpha)}$, defined over the sequence spaces c_0 and c, respectively. We proved that the sequence spaces $c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ and $c^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ are BK-spaces linearly isomorphic to c_0 and c, respectively, and obtained the Schauder basis of the spaces $c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ and $c^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$. As a result, we realised that $c_0^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ and $c^\lambda(\hat{\mathcal{F}}\Delta^{(\alpha)})$ are generalization of the spaces $c_0^\lambda(\hat{\mathcal{F}})$ and $c^\lambda(\hat{\mathcal{F}})$, respectively, defined by Das and Hazarika (Das and Hazarika, 2017). As a future scope, one may obtain duals and matrix transformation related results associated with these newly defined sequence spaces.

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