Dera Natung Government College Research Journal
Volume 9 Issue 1, pp. 1-12, January-December 2024

ISSN (Print): 2456-8228
ISSN (Online): 2583-5483

Research Article

Deferred d-statistical boundedness of order α in metric spaces

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Cite as: Aral, N.D., Kandemir, H.Ş., Et, M. (2024). Deferred d-statistical boundedness of order α in metric spaces, Dera Natung Government College Research Journal, 9, 1-12.
https://doi.org/10.56405/dngcrj.2024.09.01.01
Received on: 18.12.2023,
Revised on: 17.03.2024,
Accepted on: 09.07.2024,
Available online: 30.12.2024

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Abstract: In this study, we introduce the concept of deferred d-statistically bounded sequences of order α and give some relations betwen deferred d-statistically bounded sequences of order α and deferred d-strongly p-Cesàro summable sequences of order α in a general metric space.

Keywords: Deferred statistical convergence, Statistical boundedness, Metric space.

MSC 2020: 40A05, 40C05, 46A45.

1. Introduction

Let w be the set of all sequences of real or complex numbers. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. This sequence spaces is Banach spaces with the usual norm

$$\|x\|_\infty = \sup|x_i|.$$

where $i \in \mathbb{N} = \{1,2,\ldots\}$.

The idea of statistical convergence was given by Zygmund **(Zygmund, 1979)** in the first edition of his monograph puplished in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus **(1951)** and Fast **(1951)** and later reintroduced by Schoenberg **(1959)**. Over the years and under different names statistical convergence was discussed in the theory of Fourier analysis, Ergodic theory, Number theory, Measure theory, Trigonometric series, Turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Akbaş and Işık **(2020)**, Alotaib and Mursaleen **(2014)**, Bhardwaj and Bala **(2007)**, Bilalov and Nazarova **(2015a; 2015b; 2016)**, Bilalov and Sadigova **(2015)**, Çakallı **(1995; 2019)**, Çakallı and Kapan **(2017)**, Çakallı et al. **(2015)**, Caserta et al. **(2011)**, Çınar et al. **(2013)**, Çolak **(2010)**, Connor **(1988)**, Et et al. **(2006; 2014; 2016)**, Fridy **(1985)**, Fridy and Orhan **(1993)**, Işık and Akbaş **(2015; 2017)**, Işık and Et **(2015)**, Maio and Kočinac **(2008)**, Mursaleen **(2012)**, Salat **(1980)**, Savaş **(2017)**, Şengül **(2017a; 2017b)**, Şengül and Et **(2017)**, Şengül et al. **(2019)** and many others.

The idea of statistical convergence depends upon the density of subsets of the set $\mathbb{N}$ of natural numbers. The natural density of a subset $\mathbb{E}$ of $\mathbb{N}$ is defined by

$$\delta(\mathbb{E}) = \lim_{n\to\infty} \frac{1}{n} |k \leqslant n : k \in \mathbb{E}|, \text{ provided that the limit exists,}$$

where the vertical bars denote the cardinality of the enclosed set. In recent days natural density is being used in the study of topological coverings **(Bal & Rakshit, 2023)**.

A sequence $x = (x_i)$ is said to be statistically convergent to L if for every $\varepsilon > 0$,

$$\delta(\{i \in \mathbb{N} : |x_i - L| \geq \varepsilon\}) = 0.$$

The concept of statistical boundedness was given by Fridy and Orhan **(1997)** as follows:

A real number sequence x is statistically bounded if there is a real number B such that $\delta(\{i : |x_i| > B\}) = 0$.

In 1932, Agnew **(1932)** defined the deferred Cesaro mean $D_{c,e}$ of the sequence $x = (x_i)$ by

$$\left(D_{c,e}x\right)_t = \frac{1}{e(t) - c(t)} \sum_{i=c(t)+1}^{e(t)} x_i$$

where $c(t)$ and $e(t)$ are sequences of non-negative integers satisfying

$$c(t) < e(t)\ ,\ \lim_{t\to\infty} e(t) = +\infty \text{ and } \lim_{t\to\infty} \left(e(t) - c(t)\right) = +\infty. \quad (1)$$

We denote the set of all such (c, e) pairs by Λ. Some restrictions will be applied on (c, e) if necessary.

Let $S \subset \mathbb{N}$, and denote the set $\{s : c(t) < s \leq e(t),\ s \in S\}$ by $S_{c,e}(t)$. Deferred density of S is defined by

$$\delta_{c,e}(S) = \lim_{t\to\infty} \frac{1}{(e(t)-c(t))} \left|S_{c,e}(t)\right| \tag{2}$$

whenever the limit exists (finite or infinite).

Let X be a sequence space.

(i). A sequence space X with metric d is said to be normal (or solid) if $(x_i) \in X$ and (y_i) is sequence such that $d(y_i, a) \leq d(x_i, a)$ implies $(y_i) \in X$,

(ii). *Monotone* if it contains the canonical preimages of all its stepspaces,

(iii). *Symmetric,* if $(x_i) \in X$ implies $(x_{\pi(i)}) \in X$, where π is a permutation of $\mathbb{N}$.

2. Deferred d-statistical boundedness of order α

In this section, we introduce deferred d-statistical boundedness of order α in metric spaces. The results which we obtained in this study are much more general than those obtained by Et and Karataş **(2019)** and Şengul et al. **(2019)**.

Definition 2.1. **(Et, Çınar, and Şengül, 2019)** *Let (X, d) be a metric space, $(c, e) \in \Lambda$ and $0 < \alpha \leq 1$ be given. A metric valued sequence $x = (x_i)$ is said to be deferred d −statistically convergent of order α (or $S_{c,e}^{d,\alpha}$-convergent) if there is a real number $a \in X$ such that*

$$\lim_{t\to\infty} \frac{1}{(e(t)-c(t))^\alpha} \left|\{c(t) < i \leqslant e(t): d(x_i, a) \geq \varepsilon\}\right| = 0. \tag{3}$$

In this case we can find a real number N_ε such that

$$\left|\{c(t) < i \leqslant e(t): x_i \notin B_\epsilon(a)\}\right| \leq N_\varepsilon,$$

where $B_\epsilon(a) = \{ x \in X: d(x, a) < \varepsilon\}$ is the open ball of radius ε and center a. In this case we write $S_{c,e}^{d,\alpha} - \lim x_i = a$. The set of all deferred d-statistically convergent sequences of order α will be denoted by $S_{c,e}^{d,\alpha}$. If $\alpha = 1$, then deferred d-statistical convergence of order α coincides with deferred d-statistical convergence of sequences of real numbers which were introduced by Et et al. **(2019)**. *If $e(t) = t$, $c(t) = 0$, for all $t \in \mathbb{N}$ and $\alpha = 1$, then deferred d-statistical convergence of order α reduces to d-statistical convergence which were introduced by* Küçükaslan **(2014)**.

The deferred d-statistical convergence of order α is well defined for $0 < \alpha \leq 1$, but it is not well defined for $\alpha > 1$ in general.

Definition 2.2. *Let (X,d) be a metric space, $(c,e) \in \Lambda$ and $0 < \alpha \leq 1$ be given. A metric valued sequence $x = (x_i)$ is said to be deferred d-statistically bounded of order α (or $BS_{c,e}^{d,\alpha}$-bounded) if there is a real number $a \in X$ and a real number M such that*

$$\lim_{t\to\infty} \frac{1}{(e(t)-c(t))^{\alpha}} |\{c(t) < i \leqslant e(t): d(x_i, a) \geq M\}| = 0.$$

The set of all deferred d-statistically bounded sequences of order α will be denoted by $BS_{c,e}^{d,\alpha}$. If $e(t) = t$, $c(t) = 0$, then deferred d-statistical boundedness of order α coincides with d-statistical boundedness of order α of sequences of real numbers which were introduced by Kayan et al. **(2018)**. *If $e(t) = t$, $c(t) = 0$ and $\alpha = 1$, then deferred d-statistical boundedness of order α reduces to d-statistical boundedness which were introduced by* Küçükaslan and Deger **(2012)**.

Theorem 2.3. *Every bounded sequence is deferred d-statistically bounded of order α in a metric space, but the converse is not true.*

Proof. If $x = (x_i)$ is a bounded sequence, then for an arbitrary $x \in X$ there is $M > 0$ such that $d(x_i, a) < M$ for all $k \in \mathbb{N}$. Hence

$$\lim_{t\to\infty} \frac{1}{(e(t)-c(t))^{\alpha}} |\{c(t) < i \leqslant e(t): d(x_i, a) \geq M\}| = 0,$$

since $\{c(t) < i \leqslant e(t): d(x_i, a) \geq M\} = \phi$, for all $i \in \mathbb{N}$ and $\alpha \in (0,1]$. To show the converse part of the inclusion, choose $e(t) = t$, $c(t) = 0$, $X = \mathbb{R}$, $d(x,y) = |x-y|, \alpha = 1$ and define a sequence $x = (x_i)$ by

$$x_i = \begin{cases} i, & i = n^2, \\ (-1)^i, & \text{otherwise}, \end{cases}.$$

It is clear that $x = (x_i)$ is not bounded, but it is d-statistically bounded.

Theorem 2.4. *If the sequence $\left(\frac{c(t)}{e(t)-c(t)}\right)$ is bounded, then every d-statistically bounded sequence of order α is deferred d-statistically bounded of order α in a metric space.*

Proof. If $x = (x_i)$ be a statistically bounded sequence of order α, then for an arbitrary $x \in X$ there is $M > 0$ such that

$$\lim_{t\to\infty} \frac{1}{t^{\alpha}} |\{i \leqslant t: d(x_i, a) \geq M\}| = 0.$$

Since $\{c(t) < i \leqslant e(t): d(x_i,a) \geq M\} \subset \{i \leqslant e(t): d(x_i,a) \geq M\}$ for all $i \in \mathbb{N}$, for $\alpha \in (0,1]$, we can write

$$\frac{1}{(e(t)-c(t))^\alpha}\left|\{c(t) < i \leqslant e(t): d(x_i,a) \geq M\}\right|$$

$$\leqslant \left(\frac{e(t)}{e(t)-c(t)}\right)^\alpha \frac{1}{(e(t))^\alpha}\left|\{i \leqslant e(t): d(x_i,a) \geq M\}\right|$$

$$= \left(1+\frac{c(t)}{e(t)-c(t)}\right)^\alpha \frac{1}{(e(t))^\alpha}\left|\{i \leqslant e(t): d(x_i,a) \geq M\}\right|.$$

Taking limit as $t \to \infty$, we get the sequence $x = (x_i)$ is deferred d-statistically bounded of order α.

Theorem 2.5. *Every deferred d-statistically convergent sequence of order α is deferred d-statistically bounded of order α, but the converse is not true.*

Proof. Let $x = (x_i)$ be a deferred d-statistically convergent sequence of order α and $\varepsilon > 0$ be given. Then there exists $a \in X$ such that

$$\lim_{t\to\infty}\frac{1}{(e(t)-c(t))^\alpha}\left|\{c(t) < i \leqslant e(t): d(x_i,a) \geq \varepsilon\}\right| = 0.$$

For any $\varepsilon > 0$ and a real number M, we have

$$\{c(t) < i \leqslant e(t): d(x_i,a) \geq M\} \subset \{c(t) < i \leqslant e(t): d(x_i,a) \geq \varepsilon\}$$

and so

$$\left|\{c(t) < i \leqslant e(t): d(x_i,a) \geq M\}\right| \leq \left|\{c(t) < i \leqslant e(t): d(x_i,a) \geq \varepsilon\}\right|.$$

Taking limit $t \to \infty$, we get

$$\lim_{t\to\infty}\frac{1}{(e(t)-c(t))^\alpha}\left|\{c(t) < i \leqslant e(t): d(x_i,a) \geq M\}\right| = 0.$$

To show the converse part of the inclusion, choose $e(t) = t$, $c(t) = 0$, $X = \mathbb{R}$, $d(x,y) = |x-y|, \alpha = 1$ and define a sequence $x = (x_i)$ by

$$x_i = \begin{cases} 1, & i = 2n \\ -1, & i \neq 2n \end{cases}, i,n \in \mathbb{N}.$$

It is clear that $x=(x_i)$ is not deferred d-statistically convergent but, it is deferred d-statistically bounded.

Theorem 2.6. *Let (X,d) be a metric space and $(c,e),(c',e')\in\Lambda$ such that $c'(t)<c(t)<e(t)<e'(t)$ for all $t\in\mathbb{N}$ and let α and β be two real numbers such that $0<\alpha\le\beta\le 1$.*

(i). *Let*

$$\lim_{t\to\infty}\inf\frac{(e(t)-c(t))^{\alpha}}{(e'(t)-c'(t))^{\beta}}>0. \tag{4}$$

If a sequence $x=(x_i)$ is $BS_{c',e'}^{d,\beta}$ −bounded, then it is $BS_{c,e}^{d,\alpha}$ −bounded.

(ii). *Suppose that the inequality (4) is satisfied. Then if a sequence $x=(x_i)$ is $S_{c',e'}^{d,\beta}$-convergent, then it is $BS_{c,e}^{d,\alpha}$-bounded.*

(iii). *Let*

$$\lim_{t\to\infty}\inf\frac{(e'(t)-c'(t))^{\beta}}{(e(t)-c(t))^{\alpha}}=1. \tag{5}$$

If a sequence $x=(x_i)$ is $BS_{c,e}^{d,\alpha}$-bounded, then it is $BS_{c',e'}^{d,\beta}$-bounded.

Proof. (i) Suppose that (4) is satisfied. For given $\varepsilon>0$ we have

$$\{c'(t)<i\leqslant e'(t): d(x_i,a)\ge M\}\supseteq\{c(t)<i\leqslant e(t): d(x_i,a)\ge M\}$$

and so

$$\frac{1}{(e'(t)-c'(t))^{\beta}}\left|\{c'(t)<i\leqslant e'(t): d(x_i,a)\ge M\}\right|$$

$$\ge\frac{(e(t)-c(t))^{\alpha}}{(e'(t)-c'(t))^{\beta}}\frac{1}{(e(t)-c(t))^{\alpha}}\left|\{c(t)<i\leqslant e(t): d(x_i,a)\ge M\}\right|$$

for all $t\in\mathbb{N}$. Now taking the limit as $t\to\infty$ in the last inequality and using (4), we get $x=(x_i)$ is $BS_{c,e}^{d,\alpha}$-bounded.

(ii) Proof is similar to that of (i).

(iii) Omitted.

Theorem 2.7. *Let (X,d) be a metric space and $(c,e),(c',e')\in\Lambda$ be four sequences such that $c'(t)<c(t)<e(t)<e'(t)$ for all $t\in\mathbb{N}$ and let α and β two real numbers such that* $0<\alpha\leq\beta\leq 1$.

(i). *Suppose that the inequality* (4) *is satisfied. Then if a sequence $x=(x_i)$ is $S_{c',e'}^{d,\beta}$-convergent, then it is $S_{c,e}^{d,\alpha}$-convergent.*

(ii). *Suppose that the inequality* (5) *is satisfied. Then if a sequence $x=(x_i)$ is $S_{c,e}^{d,\alpha}$-convergent, then it is $S_{c',e'}^{d,\beta}$-convergent.*

Proof. Omitted.

Theorem 2.8. (i) $BS_{c,e}^{d,\alpha}$ *is not symmetric,*

(ii) $BS_{c,e}^{d,\alpha}$ *is normal and hence monotone.*

(iii) $BS_{c,e}^{d,\alpha}$ *is a sequence algebra.*

Proof. (i) Let $x=(x_i)=(1,0,0,2,0,0,0,0,3,0,0,0,0,0,0,4,\dots)\in BS_{c,e}^{d,\alpha}$. Let $y=(y_i)$ be a rearrangement of (x_i), which is defined as follows:

$$(y_i)=(x_1,x_2,x_4,x_3,x_9,x_5,x_{16},x_6,x_{25},x_7,x_{36},x_8,x_{49},x_{10},\dots)=(1,0,2,0,3,0,4,0,5,0,6,0,7,0,\dots).$$

Clearly, for any $M>0$, $\delta_{c,e}(\{i:|y_i|>M\})\neq 0$, in the special case $e(t)=t, c(t)=0$, $X=\mathbb{R}, d(x,y)=|x-y|$ and $\alpha=1$.

(ii) Let $x=(x_i)\in BS_{c,e}^{d,\alpha}$ and $y=(y_i)$ be a sequence such that $d(y_i,a)\leq d(x_i,a)$ for all $i\in\mathbb{N}$. Since $x\in BS_{c,e}^{d,\alpha}$ there exists a number M such that $\delta_{c,e}^{\alpha}(\{i:d(x_i,a)>M\})=0$. Clearly $y\in BS_{c,e}^{d,\alpha}$ as $\{i:d(y_i,a)>M\}\subset\{i:d(x_i,a)>M\}$. So $BS_{c,e}^{d,\alpha}$ is normal. It is well known that every normal space is monotone, so $BS_{c,e}^{d,\alpha}$ is monotone.

(iii) Let $x=(x_i), y=(y_i)\in BS_{c,e}^{d,\alpha}$. Then there exists $K,M>0$ such that $\delta_{c,e}^{\alpha}(\{i:d(x_i,a)>K\})=0$ and $\delta_{c,e}^{\alpha}(\{i:d(y_i,a)>M\})=0$. The proof follows from the following inclusion

$$\{i:d(x_iy_i,a)>KM\}\subset\{i:d(x_i,a)>K\}\cup\{i:d(y_i,a)>M\}.$$

Remark 2.9. *Using the method of* Theorem 2.8, *it can be easily shown that $S_{c,e}^{d,\alpha}$ is normal and monotone, but $S_{c,e}^{d,\alpha}$ is not symmetric.*

Theorem 2.10. *Let (X,d) be a metric space, $(c,e)\in\Lambda$ and the parameters α and β are fixed real numbers such that* $0<\alpha\leq\beta\leq 1$, *then $BS_{c,e}^{d,\alpha}\subset BS_{c,e}^{d,\beta}$ and the inclusion is strict.*

Proof. The first part of the proof is easy, so omitted.

To show the strictness of the inclusion, choose $e(t)=t, c(t)=0, X=\mathbb{R}$, $d(x,y)=|x-y|$, $a=0$ and define a sequence $x=(x_i)$ by

$$x_i=\begin{cases}\left[\sqrt{e(t)-c(t)}\right], & k=1,2,3,\ldots,\left[\sqrt{e(t)-c(t)}\right]\\ 0, & \text{otherwise}\end{cases}.$$

Then $x\in BS_{c,e}^{d,\beta}$ for $\beta\in\ (1/2,1]$, but $x\notin BS_{c,e}^{d,\alpha}$ for $x\in\left(0,\frac{1}{2}\right]$.

From Theorem 2.10 we have the following.

Corollary 2.11. *If a sequence is deferred d-statistically convergent of order α, then it is deferred d-statistically bounded.*

The proof of the following results are straightforward, so we choose to state these results without proof.

Theorem 2.12. *Let (X,d) be a metric space, $(c,e)\in\Lambda$ be two sequences and $0<\alpha\leq 1$ be given, if* $\liminf_t>1$, *then* $BS^{d,\alpha}\subseteq BS_{c,e}^{d,\alpha}$.

Theorem 2.13. *Let (X,d) be a metric space, $(c,e)\in\Lambda$ be two sequences and $0<\alpha\leq 1$ be given, if* $\limsup_t\frac{e(t)}{c(t)}<\infty$, *then* $BS_{c,e}^{d,\alpha}\subseteq BS^{d,\alpha}$.

3. Deferred d-strongly p-Cesàro summable sequences of order α

Et et al. (2019) have introduced the concept of deferred strongly d-Cesàro summability in metric spaces and give some relations on this concept. In this section we give some relations between deferred statistical boundedness of order α and deferred d-strongly p-Cesàro summable sequences of order α in metric spaces.

Definition 3.1. *Let (X,d) be a metric space, $(c,e)\in\Lambda$, $p>0$ be real numbers and $\alpha\in(0,1]$ be given. A metric valued sequence $x=(x_i)$ is said to be strongly $N_{c,e}^{d,\alpha}(p)$-summable (or deferred d-strongly p-Cesàro summable of order α) if there is a real number $a\in X$ such that*

$$\lim_{t\to\infty}\frac{1}{(e(t)-c(t))^{\alpha}}\sum_{i=c(t)+1}^{e(t)}[d(x_i,a)]^p=0.$$

In this case we write $N_{c,e}^{d,\alpha}(p)-\lim x_i=a$. *The set of all deferred* d*-strongly* p*-Cesàro summable sequences of order* α *will be denoted by* $N_{c,e}^{d,\alpha}(p)$.

In the special case $\alpha=1$, *we shall write* $N_{c,e}^{d}(p)$ *instead of* $N_{c,e}^{d,\alpha}(p)$. *If* $p=1$, *then deferred* d*-strong* p*-Cesàro summability of order* α *coincides with strong* d*-Cesàro summability of order* α *of sequences of real numbers which were introduced by* Et et al. **(2020)** *denoted by* $w^{d,\alpha}$. *If* $q(t)=t$ *and* $p(t)=0$, *then deferred strong* d*-Cesàro summability of order* α *coincides with strong* d*-Cesàro summability of order* α *of sequences of real numbers which were introduced by* Kayan et al. **(2018)** *denoted by* $w^{d,\alpha}$. *If* $a=0$, *then we shall write* $w_0^{d,\alpha}$ *instead of* $w^{d,\alpha}$.

Theorem 3.2. *Let* (X,d) *be a metric space,* $(c,e),(c',e')\in\Lambda$ *such that* $c'(t)<c(t)<e(t)<e'(t)$ *for all* $t\in\mathbb{N}$, α *and* β *be fixed real numbers such that* $0<\alpha\leq\beta\leq1$. *Then we have*

(i). *If* (4) *holds then* $N_{c',e'}^{d,\beta}(p)\subset N_{c,e}^{d,\alpha}(p)$,

(ii). *Suppose* (5) *holds and* $x=(x_i)$ *is a bounded sequence, then* $N_{c,e}^{d,\alpha}(p)\subset N_{c',e'}^{d,\beta}(p)$.

Proof. Omitted.

Theorem 3.3. *Let* (X,d) *be a metric space,* $(c,e),(c',e')\in\Lambda$ *such that* $c'(t)<c(t)<e(t)<e'(t)$ *for all* $t\in\mathbb{N}$, α *and* β *be fixed real numbers such that* $0<\alpha\leq\beta\leq1$. *Then*

(i). *Let* (4) *holds, if a sequence is strongly* $N_{c',e'}^{d,\beta}(p)$*-summable to* a, *then it is* $S_{c,e}^{d,\alpha}$*-statistically convergent to* a,

(ii). (5) *holds, if a bounded sequence is* $S_{c,e}^{d,\alpha}$*-statistically convergent to* a *then it is strongly* $N_{c',e'}^{d,\beta}(p)$*-summable to* a.

Proof. (i) Omitted.

(ii) Suppose that $S_{c,e}^{d,\alpha}-\lim x_i=a$ and $x=(x_i)$ is bounded sequence. Then there exists some $M>0$ such that $d(x_i,a)\leq M$ for all i, then for every $\varepsilon>0$, we may write

$$\frac{1}{(e'(t)-c'(t))^{\beta}}\sum_{i=c'(t)+1}^{e'(t)} d(x_i,a)=\frac{1}{(e'(t)-c'(t))^{\beta}}\sum_{i=e(t)-c(t)+1}^{e'(t)-c'(t)}[d(x_i,a)]^p+\frac{1}{(e'(t)-c'(t))^{\beta}}\sum_{i=c(t)+1}^{e(t)}[d(x_i,a)]^p$$

$$\leq\left(\frac{(e'(t)-c'(t))-(e(t)-c(t))}{(e'(t)-c'(t))^{\beta}}\right)M^p+\frac{1}{(e'(t)-c'(t))^{\beta}}\sum_{i=c(t)+1}^{e(t)}[d(x_i,a)]^p$$

$$\leq \left(\frac{(e'(t)-c'(t))-(e(t)-c(t))^{\beta}}{(e'(t)-c'(t))^{\beta}}\right) M^{p} + \frac{1}{(e'(t)-c'(t))^{\beta}} \sum_{i=c(t)+1}^{e(t)} [d(x_k,a)]^{p}$$

$$\leq \left(\frac{(e'(t)-c'(t))}{(e(t)-c(t))^{\beta}} - 1\right) M^{p} + \frac{1}{(e(t)-c(t))^{\beta}} \sum_{\substack{i=c(t)+1 \\ d(x_i,a)\geq\varepsilon}}^{e(t)} [d(x_i,a)] + \frac{1}{(e(t)-c(t))^{\beta}} \sum_{\substack{i=c(t)+1 \\ d(x_i,a)<\varepsilon}}^{e(t)} [d(x_i,a)]^{p}$$

$$\leq \left(\frac{(e'(t)-c'(t))}{(e(t)-c(t))^{\beta}} - 1\right) M^{p} + \frac{M^{p}}{(e(t)-c(t))^{\alpha}} \left|\{c(t) < i \leqslant e(t) : d(x_i,a) \geq \varepsilon\}\right| + \frac{(e(t)-c(t))}{(e(t)-c(t))^{\beta}} \varepsilon^{p}$$

for all $t \in \mathbb{N}$. Using (5), we obtain that $N_{c',e'}^{d,\beta}(p) - \lim x_i = a$, whenever $S_{c,e}^{d,\alpha} - \lim x_i = a$.

Acknowledgement: The authors are sincerely grateful to the referees for their useful remarks.

Availability of Data and Materials: No data were used to support the findings of the study.

Conflicts of Interest: The authors declare that they have no conflict of interests.

Funding: There was no funding support for this study.

Authors' Contributions: All authors contributed equally to this work. All the authors have read and approved the final version manuscript.

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