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Research Article

Some Properties of Deferred Nörlund $\mathcal{I}$-Statistical Convergence in Probability, Mean, and Distribution for Sequences of Random Variables

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Abstract: This paper investigates the concept of deferred Nörlund $\mathcal{I}$-statistical convergence in probability, mean of order r, distribution, and explores the relationships among these notions. We present a novel approach to deferred Nörlund $\mathcal{I}$-statistical convergence, which allows for a deeper understanding of the convergence behavior in various contexts. We examine the convergence properties in terms of probability, mean of order r, and distribution, providing a comprehensive analysis of their interdependencies. Our findings contribute to the field of statistical convergence theory, shedding light on the deferred Nörlund $\mathcal{I}$-statistical convergence and its connections with probability, mean of order r, and distribution.

Keywords: Probability convergence, Deferred Nörlund mean convergence, Distribution convergence, Statistical convergence.

I. Introduction and preliminaries

Fast (1951) and Schoenberg (1959) conducted pioneering research on statistical convergence. Their work laid the foundation for subsequent investigations by Rath and Tripathy (1996), Tripathy et al. (2005) and Yaying and Hazarika (2020). Central to this research is the notion of natural density, which measures the relative abundance of subsets within the set of positive integers, denoted as $\mathbb{N}$. The natural density of a subset A, represented as $\delta(A)$, is defined as the limit of $\frac{1}{n}$ multiplied by the cardinality of the set $\{k \leq n: k \in A\}$ as n approaches infinity. Expanding on the framework of statistical convergence, Kostyrko et al. (2000) introduced the concept of $\mathcal{I}$-convergence, which offers a generalization of the existing theory. Further explorations and applications of these ideas, along with the utilization of ideals, can be found in the works of (Başar, 2012, Mohiuddine et al., 2017). In a different line of research, the authors in (Das et al., 2011) introduced a novel form of convergence called $\mathcal{I}$-statistical convergence.

Suppose that (x_m) and (y_m) are the sequences of non-negative integers fulfilling

$$x_m < y_m, \forall m \in \mathbb{N} \text{ and } \lim_{x\to\infty} y_m = \infty. \tag{1.1}$$

Further, let (e_m) and (g_m) be two sequences of non-negative real numbers such that

$$\mathcal{E}_m = \sum_{n=x_m+1}^{y_m} e_n \text{ and } \mathcal{F}_m = \sum_{n=x_m+1}^{y_m} g_n. \tag{1.2}$$

The convolution of (1.2) is defined as

$$\mathcal{R}_m = \sum_{v=x_m+1}^{y_m} e_v g_{y_m-v}.$$

According to the definition given by Srivastava et al. in their 2018 study, the deferred Nörlund (DN) mean can be described as follows:

$$t_m = \frac{1}{\mathcal{R}_m} \sum_{n=x_m+1}^{y_m} e_{y_m-n} g_n y_n.$$

Let (x_m) and (y_m) be sequences that satisfy conditions (1.1), and $(e_m), (g_m)$ be sequences that satisfy condition (1.2). A sequence (Y_m) is said to exhibit deferred Nörlund statistical convergence to Y if, for any $\varepsilon > 0$, the set

$$\{n: n \leq \mathcal{R}_m, \text{ and } e_{y_m-n} g_n |Y_m - Y| \geq \varepsilon\}$$

has zero deferred Nörlund density, i.e. if

$$\lim_{m\to\infty} \frac{1}{\mathcal{R}_m} \mid \{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n |Y_m - Y| \geq \varepsilon\} \mid = 0.$$

We write it as

$$St_{DN}\lim Y_m = Y.$$

Let (x_m) and (y_m) be sequences that satisfy the conditions (1.1), and $(e_m),(g_m)$ be sequences that satisfy the conditions (1.2).

A sequence (Y_m) is called as deferred Nörlund statistically probability (or St_{DNP})-convergent to a random variable Y, if $\forall\varepsilon > 0$ and $\delta > 0$, the set

$$\{n: n \le \mathcal{R}_m \text{ and } e_{y_m-n}g_nP(|Y_m - Y| \ge \varepsilon) \ge \delta\}$$

has DN-density zero, i.e.,

$$\lim_{m\to\infty}\frac{1}{\mathcal{R}_m}\mid\{n: n \le \mathcal{R}_m \text{ and } e_{y_m-n}g_nP(|Y_m - Y| \ge \varepsilon) \ge \delta\}\mid = 0$$

or

$$\lim_{m\to\infty}\frac{1}{\mathcal{R}_m}\mid\{n: n \le \mathcal{R}_m \text{ and } 1 - e_{y_m-n}g_nP(|Y_m - Y| \le \varepsilon) \ge \delta\}\mid = 0,$$

and it is denoted as

$$St_{DNP}\lim_{m\to\infty} e_{y_m-n}g_nP(|Y_m - Y| \ge \varepsilon) = 0$$

or

$$St_{DNP}\lim_{m\to\infty} e_{y_m-n}g_nP(|Y_m - Y| \le \varepsilon) = 1.$$

In this section, our focus lies on the investigation of deferred Nörlund $\mathcal{I}$-statistical probability convergence. For a comprehensive understanding of the historical context and fundamental concepts, we refer the interested reader to the following references: (Ghosal, 2013, Eunice Jemima and Srinivasan, 2022, Raj and Jasrotia, 2023, Jena et al., 2020, Srivastava et al., 2018, Srivastava et al., 2019, Srivastava et al., 2020a, Srivastava et al., 2020b).

We will now delve into foundational concepts that are closely related to our research findings.

The concept of statistical convergence was further extended in the paper by (Kostyrko et al., 2000) through the introduction of ideals of subsets of the set $\mathbb{N}$. An ideal on $\mathbb{N}$ is defined as a non-empty family of sets $\mathcal{J} \subset \mathcal{P}(\mathbb{N})$ that satisfies the properties of heredity (i.e., $B \subset A \in \mathcal{J}$ implies $B \in \mathcal{J}$) and additivity (i.e., $A, B \in \mathcal{J}$ implies $A \cup B \in \mathcal{J}$). If $\mathcal{J} \neq \mathcal{P}(\mathbb{N})$, then the ideal $\mathcal{J}$ is referred to as a non-trivial ideal. Moreover, a non-trivial ideal $\mathcal{J}$ is considered admissible if it contains all finite subsets of $\mathbb{N}$. In the subsequent discussion, unless otherwise specified, $\mathcal{J}$ will represent an admissible ideal.

For any ideal $\mathcal{J} \subset \mathcal{P}(\mathbb{N})$, there exists a corresponding filter on N denoted by $\mathcal{F}(\mathcal{J}) = \{M \subset \mathbb{N}: \exists A \in \mathcal{J}: M = \mathbb{N} \setminus A\}$. This filter is referred to as the filter associated with the ideal $\mathcal{J}$.

Definition 1.1: Let $\mathcal{J}$ be an admissible ideal on $\mathbb{N}$ and $x = (x_k)$ be a real sequence. We say that the sequence x is $\mathcal{J}$-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N}: |x_k - L| \geq \varepsilon\} \in \mathcal{J}$.

Take for $\mathcal{J}$ the class $\mathcal{J}_f$ of all finite subsets of $\mathbb{N}$. Then $\mathcal{J}_f$ is a non-trivial admissible ideal and $\mathcal{J}_f$-convergence coincides with the usual convergence.

We also recall that the concept of $\mathcal{J}$-statistically convergent was studied in (Das et al., 2011).

Definition 1.2: A sequence (x_k) is said to be $\mathcal{J}$-statistically convergent to L if for each $\varepsilon > 0$ and $\delta > 0$.

$$\left\{n \in \mathbb{N}: \frac{1}{n}|\{k \leq n: |x_k - L| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{J}.$$

In this case, L is called $\mathcal{J}$-statistical limit of the sequence (x_k) and we write $\mathcal{J} - st - \lim_{k\to\infty} x_k = L$.

II. Main Results

Based on preliminary examination, our objective is to extend the investigations conducted by Srivastava et al. (2020) and explore diverse aspects of $\mathcal{J}$-statistical convergence for sequences of random variables and sequences of real numbers through deferred Nörlund summability mean. Our analysis focuses on examining multiple results that establish the relationship by utilizing fundamental limit concepts of sequences of random variables.

Definition 2.1: Let (x_m) and (y_m) be sequences that satisfy the conditions (1.1), and $(e_m), (g_m)$ be sequences that satisfy the conditions (1.2). A sequence (Y_m) is called as deferred Nörlund $\mathcal{J}$-statistically (or $St_{DN}(\mathcal{J})$)-convergent to Y, if $\forall \varepsilon > 0$ and $\delta > 0$

$$\left\{m \in \mathbb{N}: \frac{1}{\mathcal{R}_m} \left|\left\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n |Y_m - Y| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{J}.$$

We write it as $Y_m \xrightarrow{St_{DN}(\mathcal{J})} Y$ or $\mathcal{J} - St_{DN}\lim_{m\to\infty} Y_m = Y$.

Definition 2.2: Let (x_m) and (y_m) be sequences that satisfy the conditions (1.1), and $(e_m), (g_m)$ be sequences that satisfy the conditions (1.2). A sequence (Y_m) is called as deferred Nörlund $\mathcal{J}$-statistically probability (or $St_{DNP}(\mathcal{J})$)-convergent to a random variable Y, if $\forall \varepsilon, \delta > 0$ and $\eta > 0$

$$\left\{m \in \mathbb{N}: \frac{1}{\mathcal{R}_m} \left|\left\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n P(|Y_m - Y| \geq \varepsilon) \geq \delta\right\}\right| \geq \eta\right\} \in \mathcal{J},$$

or equivalently,

$$\left\{m \in \mathbb{N}: \frac{1}{\mathcal{R}_m} \left|\left\{n: n \leq \mathcal{R}_m \text{ and } 1 - e_{y_m-n} g_n P(|Y_m - Y| < \varepsilon) \geq \delta\right\}\right| \geq \eta\right\} \in \mathcal{J}.$$

In any case, we will denote them as $Y_m \xrightarrow{St_{DNP}(\mathcal{J})} Y$ or $\mathcal{J} - St_{DNP} - \lim_{m\to\infty} e_{y_m-n} g_n P(|Y_m - Y| \geq \varepsilon) = 0$. The class of all deferred Nörlund $\mathcal{J}$-statistically probability convergent sequences of random variables will be denoted by $St_{DNP}(\mathcal{J})$.

Theorem 2.1: Suppose that (Y_m) is sequence of random variables and consider two random variables Y and Z. If $Y_m \xrightarrow{St_{DNP}(\mathcal{J})} Y$ and $Y_m \xrightarrow{St_{DNP}(\mathcal{J})} Z$, then $P(Y = Z) = 1$.

Proof: Let $\varepsilon, \delta > 0$ and $0 < \eta < 1$, then

$$A = \left\{m \in \mathbb{N}: \frac{1}{\mathcal{R}_m} \left|\left\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n P\left(|Y_m - Y| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right| < \frac{\eta}{3}\right\} \in F(\mathcal{J}),$$

$$B = \left\{m \in \mathbb{N}: \frac{1}{\mathcal{R}_m} \left|\left\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n P\left(|Y_m - Z| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right| < \frac{\eta}{3}\right\} \in F(\mathcal{J}).$$

Since $A \cap B \in F(I)$ and $\emptyset \notin F(\mathcal{J})$ implies that $A \cap B \neq \emptyset$. Now let $s \in A \cap B$. Then

$$\frac{1}{\mathcal{R}_m} \left|\left\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n P\left(|Y_s - Y| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right| < \frac{\eta}{3}$$

and

$$\frac{1}{\mathcal{R}_m}\left|\left\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n P\left(|Y_s - Z| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right| < \frac{\eta}{3}.$$

This implies,

$$\frac{1}{\mathcal{R}_m}\left|\left\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n P\left(|Y_s - Y| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2} \text{ or } e_{y_m-n} g_n P\left(|Y_s - Z| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right| < \eta < 1.$$

Therefore, there exists any $n \leq \mathcal{R}_m$ such that $e_{y_m-n} g_n P\left(|Y_s - Y| \geq \frac{\varepsilon}{2}\right) < \frac{\delta}{2}$ and $e_{y_m-n} g_n P\left(|Y_s - Z| \geq \frac{\varepsilon}{2}\right) < \frac{\delta}{2}$. Hence,

$$e_{y_m-n} g_n P\left(|\mathrm{Y} - Z| \geq \frac{\varepsilon}{2}\right) \leq e_{y_m-n} g_n P\left(|Y_s - Y| \geq \frac{\varepsilon}{2}\right) + e_{y_m-n} g_n P\left(|Y_s - Z| \geq \frac{\varepsilon}{2}\right) < \delta < 1.$$

It means

$$P(Y = Z) = 1.$$

Theorem 2.2: Suppose that (Y_m) is sequence of random variables. If $Y_m \xrightarrow{St_{DNP}(\mathcal{I})} y$, then $Y_m^2 \xrightarrow{St_{DNP}(\mathcal{I})} y^2$.

Proof: If $Y_m \xrightarrow{St_{DNP}(\mathcal{I})} 0$, then $Y_m^2 \xrightarrow{St_{DNP}(\mathcal{I})} 0$. Here, we see that

$$\left\{m \in \mathbb{N}: \frac{1}{\mathcal{R}_m}\left|\left\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n P(|Y_m^2 - 0| \geq \varepsilon) \geq \delta\right\}\right| \geq \eta\right\}$$
$$= \left\{m \in \mathbb{N}: \frac{1}{\mathcal{R}_m}\left|\left\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n P\left(|Y_m - 0| \geq \sqrt{\varepsilon}\right) \geq \delta\right\}\right| \geq \eta\right\} \in \mathcal{I}.$$

Now, take $Y_m^2 = (Y_m - y)^2 + 2y(Y_m - y) + y^2$. Thus, $Y_m^2 \xrightarrow{St_{DNP}(\mathcal{I})} y^2$.

Theorem 2.3: Suppose that (Y_m) and (Z_m) are sequences of random variables and consider two random variables Y and Z. Then, the following assertions are satisfied:

1) If $Y_m \xrightarrow{St_{DNP}(\mathcal{I})} y$ and $Z_m \xrightarrow{St_{DNP}(\mathcal{I})} z$, then $Y_m Z_m \xrightarrow{St_{DNP}(\mathcal{I})} yz$,
2) If $Y_m \xrightarrow{St_{DNP}(\mathcal{I})} y$ and $Z_m \xrightarrow{St_{DNP}(\mathcal{I})} z$, then $\frac{Y_m}{Z_m} \xrightarrow{St_{DNP}(\mathcal{I})} \frac{y}{z}, z \neq 0$,
3) If $Y_m \xrightarrow{St_{DNP}(\mathcal{I})} Y$ and $Z_m \xrightarrow{St_{DNP}(\mathcal{I})} Z$, then $Y_m Z_m \xrightarrow{St_{DNP}(\mathcal{I})} YZ$,
4) If $Y_m \xrightarrow{St_{DNP}(\mathcal{I})} Y$ and for each $\varepsilon, \delta, \eta > 0$, then we have

$$\left\{m \in \mathbb{N}: \frac{1}{\mathcal{R}_m}\left|\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n P(|Y_m - Y| \geq \varepsilon) \geq \delta\}\right| \geq \eta\right\} \in \mathcal{I}.$$

Proof: 1) Suppose that $Y_m \xrightarrow{St_{DNP}(\mathcal{I})} y$ and $Z_m \xrightarrow{St_{DNP}(\mathcal{I})} z$. We get

$$Y_m Z_m = \frac{1}{4}\{(Y_m + Z_m)^2 - (Y_m - Z_m)^2\} \xrightarrow{St_{DNP}(\mathcal{I})} \frac{1}{4}\{(y+z)^2 - (y-z)^2\} = yz.$$

2) Assume that A and B be two events correspond $|Z_m - z| < |z|$ and $\left|\frac{1}{Z_m} - \frac{1}{z}\right| \geq \varepsilon$. We obtain

$$\left|\frac{1}{Z_m} - \frac{1}{z}\right| = \frac{|Z_m - z|}{|zZ_m|} = \frac{|Z_m - z|}{|z| \cdot |z + (Z_m - z)|} \leq \frac{|Z_m - z|}{|z| \cdot |(|z| - |Z_m - z|)|}.$$

If the events A and B occurs at same time, then

$$|Z_m - z| \geq \frac{\varepsilon |z^2|}{1 + \varepsilon |z|}.$$

Further, let $\varepsilon_0 = \varepsilon |z|^2 / (1 + \varepsilon |z|)$ and C be the event such that $|Z_m - z| \geq \varepsilon_0$. Thus

$$AB \subseteq C \Rightarrow P(B) \leq P(C) + P(A^c)$$

Thus,

$$\begin{aligned}
&\left\{m \in \mathbb{N}: \frac{1}{\mathcal{R}_m}\left|\left\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n P\left(\left|\frac{1}{Z_m} - \frac{1}{z}\right| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \eta\right\} \\
&\subseteq \left\{m \in \mathbb{N}: \frac{1}{\mathcal{R}_m}\left|\left\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n P(|Z_m - z| \geq \varepsilon_0) \geq \frac{\delta}{2}\right\}\right| \geq \eta\right\} \\
&\cup \left\{m \in \mathbb{N}: \frac{1}{\mathcal{R}_m}\left|\left\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n P(|Z_m - z| \geq |z|) \geq \frac{\delta}{2}\right\}\right| \geq \eta\right\}.
\end{aligned}$$

Therefore, $\frac{1}{Z_m} \xrightarrow{St_{DNP}(\mathcal{I})} \frac{1}{z}$. Hence, we write $\frac{Y_m}{Z_m} \xrightarrow{St_{DNP}(\mathcal{I})} \frac{y}{z}, z \neq 0$.

3. Suppose that $Y_m \xrightarrow{St_{DNP}(\mathcal{I})} Y$ and X be a random variable such that $Y_m X \to YX$. Since X is a random variable such that $\forall \alpha > 0, \exists \delta > 0$ and $e_{y_m-n} g_n P(|X| > \alpha) \leq \frac{\delta}{2}$. Then, for any $\varepsilon > 0$

$$e_{y_m-n}g_nP(|Y_mX-YX|\geq\varepsilon)= e_{y_m-n}g_nP(|Y_m-Y||X|\geq\varepsilon,|Z|>\alpha)$$
$$+e_{y_m-n}g_nP(|Y_m-Y||X|\geq\varepsilon,|Z|\leq\alpha)\leq\frac{\delta}{2}$$
$$+e_{y_m-n}g_nP\left(|Y_m-Y|\geq\frac{\varepsilon}{\alpha}\right)$$

which implies,

$$\left\{m\in\mathbb{N}:\frac{1}{\mathcal{R}_m}\left|\left\{n:n\leq\mathcal{R}_m \text{ and } e_{y_m-n}g_nP(|Y_mX-YX|\geq\varepsilon)\geq\delta\right\}\right|\geq\eta\right\}$$
$$\subseteq\left\{m\in\mathbb{N}:\frac{1}{\mathcal{R}_m}\left|\left\{n:n\leq\mathcal{R}_m \text{ and } e_{y_m-n}g_nP\left(|Y_m-y|\geq\frac{\varepsilon}{\alpha}\right)\geq\frac{\delta}{2}\right\}\right|\geq\eta\right\}.$$

Therefore,

$$(Y_m-Y)(Z_m-Z)\xrightarrow{St_{DNP}(\mathcal{I})}0.$$

Thus,

$$Y_mZ_m\xrightarrow{St_{DNP}(\mathcal{I})}YZ.$$

4. Suppose that (x_m) and (y_m) be two non-negative sequences such that

$$\frac{1}{\mathcal{R}_m}\left|\left\{n:n\leq\mathcal{R}_m \text{ and } e_{y_m-n}g_nP\left(|Y_m-Y|\geq\frac{\varepsilon}{2}\right)\geq\frac{\delta}{2}\right\}\right|<\frac{\eta}{2}.$$

and

$$\left\{m\in\mathbb{N}:\frac{1}{\mathcal{R}_m}\left|\left\{n:n\leq\mathcal{R}_m \text{ and } e_{y_m-n}g_nP\left(|Y_m-Y|\geq\frac{\varepsilon}{2}\right)\geq\frac{\delta}{2}\right\}\right|<\frac{\eta}{2}\right\}\in F(\mathcal{I}).$$

Now,

$$\left\{m\in\mathbb{N}:\frac{1}{\mathcal{R}_m}\left|\left\{n:n\leq\mathcal{R}_m \text{ and } e_{y_m-n}g_nP(|Y_m-Y|\geq\varepsilon)\geq\delta\right\}\right|\geq\eta\right\}$$
$$\subseteq\left\{m\in\mathbb{N}:\frac{1}{\mathcal{R}_m}\left|\left\{n:n\leq\mathcal{R}_m \text{ and } e_{y_m-n}g_nP\left(|Y_m-Y|\geq\frac{\varepsilon}{2}\right)\geq\frac{\delta}{2}\right\}\right|\geq\frac{\eta}{2}\right\}\in\mathcal{I},$$

which implies that

$$\left\{m \in \mathbb{N}: \frac{1}{\mathcal{R}_m}\left|\left\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n P(|Y_m - Y| \geq \varepsilon) \geq \delta\right\}\right| \geq \eta\right\} \in \mathcal{J}.$$

Theorem 2.4: Suppose that $f: \mathbb{R} \to \mathbb{R}$ is uniform continuous on $\mathbb{R}$ and $Y_m \xrightarrow{St_{DNP}(\mathcal{J})} Y$. Then $f(Y_m) \xrightarrow{St_{DNP}(\mathcal{J})} f(Y)$.

Proof: Let us consider a random variable Y such that for each $\delta > 0, \exists \beta \in \mathbb{R}$ such that $P(Y > \beta) \leq \delta/2$. Since, f is uniformly continuous on $\beta, \forall \varepsilon > 0, \exists \delta_0$ such that

$$|f(y_m) - f(y)| < \varepsilon \text{ whenever } |y_m - y| < \delta_0.$$

Thus,

$$\begin{aligned} P(|f(Y_m) - f(Y)| \geq \varepsilon) \quad &\leq P(|Y_m - Y| \geq \delta_0) + P(|Y > \beta|) \\ &\leq P(|Y_m - Y| \geq \delta_0) + \delta/2. \end{aligned}$$

By considering the definition of $St_{DNP}(\mathcal{J})$-convergence, we can deduce the following:

$$\begin{aligned} \{n: n \quad &\leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n P(|f(Y_m) - f(Y)| \geq \varepsilon) \geq \delta\} \\ &\subseteq \left\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n P(|Y_m - Y| \geq \delta_0) < \frac{\delta}{2}\right\}. \end{aligned}$$

Therefore, we obtain

$$\left\{m \in \mathbb{N}: \frac{1}{\mathcal{R}_m}\left|\left\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n P(|f(Y_m) - f(Y)| \geq \varepsilon) \geq \delta\right\}\right| \geq \eta\right\} \in \mathcal{J}.$$

Definition 2.3: A sequence (Y_m) is $\mathcal{J}$-statistically r^{th} mean convergent (MC) to a random variable Y, where $Y: S \to \mathbb{R}$ if,

$$\left\{m \in \mathbb{N}: \frac{1}{m}\left|\{n: n \leq m \text{ and } E(|Y_m - Y|^r \geq \varepsilon)\}\right| \geq \delta\right\} \in \mathcal{J},$$

for any $\varepsilon, \delta > 0$.

We write it as $Y_m \xrightarrow{St_{MC}(\mathcal{J})} Y$.

Definition 2.4*:* Suppose that (x_m) and (y_m) are the sequences fulfilling conditions (1.1) and $(e_m), (g_m)$ are sequences satisfying (1.2). A sequence (Y_m) is said to be deferred Nörlund $\mathcal{J}$-statistically $r^{th} (r \geq 1)$ mean convergent to $Y (Y: S \to \mathbb{R})$, if for $\varepsilon, \delta > 0$,

$$\left\{m \in \mathbb{N}: \frac{1}{\mathcal{R}_m}\left|\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m-n} g_n E(|Y_m - Y|^r \geq \varepsilon)\}\right| \geq \delta\right\} \in \mathcal{I}.$$

It is denoted as $Y_m \xrightarrow{St_{DNM}(\mathcal{I})} Y$ or $\mathcal{I} - St_{DNM} - \lim_{m\to\infty} E(|Y_m - Y|^r) = 0$.

Theorem 2.5. Let $\mathcal{I} - St_{DNM} - \lim_{m\to\infty} E(|Y_m - Y|^r) = 0$ for $r \geq 1$, then $\mathcal{I} - St_{DNP} - \lim_{m\to\infty} P(|Y_m - Y| \geq \varepsilon) = 0$.

Proof: For every $\varepsilon > 0$, we have from Markov's inequality

$$\begin{aligned} \mathcal{I} - St_{DNP} - \lim_{m\to\infty} P(|Y_m - Y| \geq \varepsilon) &= \mathcal{I} - St_{DNP} - \lim_{m\to\infty} P(|Y_m - Y|^r \geq \varepsilon^r), (r \geq 1) \\ &\leq \mathcal{I} - St_{DNM} - \lim_{m\to\infty} \frac{E(|Y_m - Y|^r)}{\varepsilon^r} = 0. \end{aligned}$$

From definition of statistically deferred Nörlund mean convergence

$$\mathcal{I} - St_{DNM} - \lim_{m\to\infty} E(|Y_m - Y|^r) = 0$$

it implies that

$$\mathcal{I} - St_{DNP} - \lim_{m\to\infty} P(|Y_m - Y| \geq \varepsilon) = 0.$$

Here we provide an illustrative example to demonstrate that a sequence of random variables exhibits $\mathcal{I}$-statistical probability convergence but does not exhibit $\mathcal{I}$-statistical r^{th}-mean convergence.

Example 2.1: Suppose that $x_m = 2m - 1, y_m = 4m - 1$. Also, suppose that $e_{y_m-m} = 2m$ and $g_m = 1$. Further, consider a sequence (z_m) of random variables such that

$$Y_m = \begin{cases} m, \text{ with probability } \dfrac{1}{\sqrt{m}} \\ 0, \text{ with probability } 1 - \dfrac{1}{\sqrt{m}} \end{cases}$$

Then the $\mathcal{I}$-statistically deferred Nörlund convergence of Y_m is given as

$$\left\{m \in \mathbb{N}: \frac{1}{2m}\left|\{n: n \leq \mathcal{R}_m \text{ and } 2mP(|Y_m - 0| \geq \varepsilon)\}\right| \geq \delta\right\} \in \mathcal{I}.$$

However, $\mathcal{I}$-statistically deferred Nörlund mean convergence, for $r \geq 1$, is

$$\left\{m \in \mathbb{N}: \frac{1}{2m} |\{n: n \leq \mathcal{R}_m \text{ and } 2mE(|Y_m - 0|^r)\}| \geq \delta\right\} \notin \mathcal{I}.$$

This implies that the sequence (Y_m) is $\mathcal{I} - St_{DNP}$-convergent but not $\mathcal{I} - St_{DNM}$-convergent.

Throughout the paper $\left(F_{Y_m}(y)\right)$ is the sequence of distribution functions of (Y_m) and $F_Y(y)$ is the distribution function of Y.

Definition 2.5: The sequence $\left(F_{Y_m}(y)\right)$ is called as $\mathcal{I}$-statistically distribution convergent (or $St_{DC}(\mathcal{I})$), if there exists $F_Y(y)$ of random variable Y such that for all $\varepsilon, \delta > 0$,

$$\left\{m \in \mathbb{N}: \frac{1}{m} \left|\left\{n: n \leq m \text{ and } \left|F_{Y_m}(y) - F_Y(y)\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

We may write this as $F_{Y_m}(y) \xrightarrow{St_{DC}(\mathcal{I})} F_Y(y)$ or $\mathcal{I} - St_{DC} - \lim_{m\to\infty} \lim_{m\to\infty} F_{Y_m}(y) = F_Y(y)$.

Definition 2.6: The sequence $\left(F_{Y_m}(y)\right)$ of distribution functions is called as deferred Nörlund $\mathcal{I}$-statistically distribution convergent (or $St_{DNDC}(\mathcal{I})$), if there exists $F_Y(y)$ of Y such that for each $\varepsilon, \delta > 0$

$$\left\{m \in \mathbb{N}: \frac{1}{\mathcal{R}_m} \left|\left\{n: n \leq \mathcal{R}_m \text{ and } e_{y_m - n} g_n \left|F_{Y_m}(y) - F_Y(y)\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we say $F_{Y_m}(y) \xrightarrow{St_{DNDC}(\mathcal{I})} F_Y(y)$ or $\mathcal{I} - St_{DNDC} - \lim_{m\to\infty} \lim_{m\to\infty} F_{Y_m}(y) = F_Y(y)$.

Theorem 2.6: Suppose that $\mathcal{I} - St_{DNP} \lim_{m\to\infty} P(|Y_m - Y| \geq \varepsilon) = 0$, then

$$\mathcal{I} - St_{DNDC} \lim_{m\to\infty} F_{Y_m}(y) = F_Y(y).$$

Proof: Suppose that $\left(F_{Y_m}(y)\right)$ is distribution functions of (Y_m), and $F_Y(y)$ be the distribution function of Y. For $i, j \in \mathbb{R}$ such that $i < j$, we have

$$(Y \leq i) = (Y_m \leq j, Y \leq i) + (Y_m \geq j, Y \leq i)$$

Further,

$$(Y_m \leq j, Y \leq i) \subseteq (Y_m \leq j)$$

which implies that

$$(Y \leq i) \subseteq (Y_m \leq j) + (Y_m \geq j, Y \leq i). \tag{2.1}$$

Let us take the probability to left hand side and right hand side of equation (2.1)

$$\begin{aligned} P(Y \leq i) \quad &\leq P\{(Y_m \leq j) + (Y_m \geq j, Y \leq i)\} \\ &\leq P(Y_m \leq j) + P(Y_m \geq j, Y \leq i) \end{aligned}$$

It means that

$$F_{Y_m}(j) \geq F_Y(i) - P(Y_m \geq j, Y \leq i). \tag{2.2}$$

If $Y_m \geq j, Y \leq i$, then $Y_m \geq j, -Y \geq -i$, so that $Y_m - Y > j - i$, that is,

$$(Y_m \geq j, Y \leq i) \subseteq (Y_m - Y > j - i) \subseteq (|Y_m - Y| > j - i)$$

This means

$$P(Y_m \geq j, Y \leq i) \leq P(|Y_m - Y| > j - i)$$

As we know that $i < j$ and $\mathcal{I} - St_{DNP} Y_m \to Y$, we obtain

$$\mathcal{I} - St_{DNP} \lim_{m\to\infty} P(Y_m \geq j, Y \leq i) = 0.$$

From (2.2) we get

$$\mathcal{I} - St_{DNDC} \lim_{m\to\infty} F_{Y_m}(j) \geq F_Y(i).$$

Similarly, if $j < a$ for any real constant a, then

$$(Y \leq j) = (Y \leq a, Y_m \leq j) + (Y > a, Y_m \leq j).$$

Consequently,

$$F_{Y_m}(j) \leq F_Y(a) + P(Y > a, Y_m \leq j)$$

and

$$\mathcal{I} - St_{DNDC} \lim_{m\to\infty} P(Y > a, Y_m \leq j) = 0.$$

Therefore, we get

$$\mathcal{I} - St_{DNDC} \lim_{m\to\infty} F_{Y_m}(j) \leq F_Y(a).$$

Thus, with $i < j < a$, we have

$$\mathcal{I} - St_{DNDC} \lim_{m\to\infty} F_{Y_m}(j) = F_Y(i).$$

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