Dera Natung Government College Research Journal
Volume 8 Issue 1, pp. 30-45, January-December 2023

ISSN (Print): 2456-8228
ISSN (Online): 2583-5483

Research Article

A New Paranormed Sequence Space Defined by Regular Bell Matrix

Murat Karakaş[a], **Muhammet Cihat Dağlı**[b,*]

[a]Department of Mathematics, Bitlis Eren University,13000-Bitlis, Turkey.
[b]Department of Mathematics, Akdeniz University,07058-Antalya, Turkey.

Cite as: Karakaş, M., Dağlı, M.C. (2023). A new paranormed sequence space defined by regular Bell matrix. Dera Natung Government College Research Journal, 8, 30-45. https://doi.org/10.56405/dngcrj.2023.08.01.03

Received on: 07.08.2023,
Revised on: 13.09.2023,
Accepted on: 13.09.2023,
Available online: 26.12.2023.

*Corresponding Author: M.C. Dağlı (mcihatdagli@akdeniz.edu.tr)

Abstract: This paper aims to construct a new paranormed sequence space by the aid of a regular matrix of Bell numbers. As well, its special duals such as $\alpha-$, $\beta-$, $\gamma-$duals are presented and Schauder basis is determined. Moreover, certain matrix classes for this space are characterized.

Keywords: Bell numbers, Paranormed sequence space, Matrix transformation, $\alpha-$, $\beta-$, $\gamma-$duals.

MSC Subject classification: 11B73, 46B45, 46A45, 40C05, 47B37, 47B07.

I. Introduction

A linear subspace of the set of all real valued sequences ω is known as a sequence space. The notations c, c_0, ℓ_∞ and ℓ_p represent the set of all convergent sequences, the set of all convergent to zero sequences, the set of all bounded sequences and the set of all sequences constituting $p-$absolutely convergent series, respectively. Also, these spaces are Banach spaces with regard to the following norms

$$\|t\|_{\ell_\infty} = \|t\|_c = \|t\|_{c_0} = \sup_{i \in \mathbb{N}} |t_i|$$

and

$$\|t\|_{\ell_p} = \left(\sum_i |t_i|^p \right)^{1/p}.$$

For brevity, here and afterwards, it means that the summation without limits runs from 0 to ∞ .

© 2023 by Dera Natung Government College, Itanagar, India.

This work is licensed under a Creative Commons Attribution 4.0 International License.

Let H be a linear topological space over $\mathbb{R}$. If there exists a subadditive function $\varsigma : H \to \mathbb{R}$ such that $\varsigma(\theta)=0$, $\varsigma(-t)=\varsigma(t)$; $|\alpha_i-\alpha|\to 0$ and $\varsigma(t_i-t)\to 0$ imply that $\varsigma(\alpha_i t_i-\alpha t)\to 0$ for all $\alpha\in\mathbb{R}$ and $t\in H$, then, H is known as paranormed space.

Assume here and after that (p_i) be a bounded sequence in $\mathbb{R}$ such that $p_i>0$, $\sup_{i\in\mathbb{N}} p_i = P$ and $S=\max\{1,P\}$. Then, we have from Maddox (Maddox, 1967) that

$$|\xi|^{p_i}\le \max\left\{1,|\xi|^S\right\} \tag{1.1}$$

for any $\xi\in\mathbb{R}$ and $i\in\mathbb{N}=\{1,2,...\}$. In what follows, suppose that $p_i^{-1}+\left(p_i^{'}\right)^{-1}=1$ with $1<\inf p_i<P<\infty$. Then, Maddox (Maddox 1968, 1988) (consult also Nakano, 1951 and Simons, 1965) defined the linear space $\ell(p)$ as

$$\ell(p)=\left\{t=(t_i)\in\omega:\sum_i |t_i|^{p_i}<\infty\right\}$$

which is a complete space paranormed by

$$\varsigma(t)=\left(\sum_i |t_i|^{p_i}\right)^{1/S}.$$

For each $i\in\mathbb{N}$, let U_i be the sequence in the ith row of an infinite matrix $U=(u_{ij})$. The sequence Ut will be used in the sequel as $U-$transform of a sequence $t=(t_j)\in\omega$ and its ith $(i\in\mathbb{N})$ entry is given by

$$(Ut)_i=\sum_j u_{ij}t_j$$

on the condition that the above series is convergent for each $j\in\mathbb{N}$. U is called as a matrix mapping from a sequence space H to a sequence space G if the sequence Ut exists and $Ut\in G$ for all $t\in H$. The impression (H,G) represents the class of all infinite matrices between the spaces H and G.

The domain of the infinite matrix U is given by the following set:

$$H_U=\{t\in\omega: Ut\in H\}.$$

Recently, there has been an increase in the body of literature that deals with creating new sequence spaces using a special limitation method with the help of matrix domain and researching their topological, algebraic features and matrix transformations. For instance, one can refer Alp (Alp, 2020), Altay and Başar (Altay and Başar, 2006), Aydın and Başar (Aydın and Başar, 2004), Candan (Candan, 2012), Candan and Güneş (Candan and Güneş, 2015), Choudary and Mishra (Choudary and Mishra, 1993), Dağlı (Dağlı, 2023), Dağlı and Yaying (Dağlı and Yaying, 2023), Et and Colak (Et and Colak, 1995), İlkhan et al. (İlkhan et al. 2019, 2020a, 2020b), Kara and Demiriz (Kara and Demiriz, 2015), Karakaya et al. (Karakaya et al., 2011, 2013), Karakaya and Simsek (Karakaya and Simsek, 2012), Kirisci and Başar (Kirisci and Başar, 2010), Malkowsky (Malkowsky, 1997), Malkowsky et al. (Malkowsky et al., 2017), Malkowsky and Savas (Malkowsky and Savas, 2004), Yaying (Yaying, 2022) and related references therein.

This paper deals with well known Bell numbers whose i^{th} entry is denoted by B_i. The i^{th} Bell number represents how many different ways a set of i elements may be divided into non-empty subsets. For instance, there are five possible divisions of the numbers $\{1,2,3\}$:

$$\{1\}\cup\{2\}\cup\{3\},\{1,2\}\cup\{3\},\{1,3\}\cup\{2\},\{1\}\cup\{2,3\},\{1,2,3\}.$$

The first few Bell numbers are $1,2,5,15,52,203,877,4140,...$ with the initial condition $B_0 = 1$. They can be defined as

$$B_{i+1} = \sum_{j=0}^{i} \binom{i}{j} B_j$$

and

$$B_i = \sum_{j=0}^{i} S_2(i,j),$$

where $S_2(i,j)$ is the Stirling numbers of the second kind, counts the set partitions of $\{1,2,3,...,i\}$ which consists of exactly j subsets or parts.

Quite recently, Karakas (Karakas, 2023) introduced a new matrix $\tilde{B} = (\tilde{b}_{ij})$ involving Bell numbers defined by

$$\tilde{b}_{ij} = \begin{cases} \binom{i}{j} \dfrac{B_j}{B_{i+1}}, & \text{if } 0 \le j \le i; \\ 0, & \text{if } j > i; \end{cases}$$

and observed that this matrix is regular. Also, he considered the sequence spaces $\ell_p(\tilde{B})$ for $1 \le p < \infty$ and $\ell_\infty(\tilde{B})$ as the set of all sequences whose $\tilde{B}-$transforms are in the spaces ℓ_p and ℓ_∞, respectively. In other words,

$$\ell_p(\tilde{B}) = \{t = (t_j) \in \omega : \tilde{B}t \in \ell_p\} \text{ for } 1 \le p < \infty$$

and

$$\ell_\infty(\tilde{B}) = \{t = (t_j) \in \omega : \tilde{B}t \in \ell_\infty\}.$$

Also, some topological properties such as giving Schauder basis, determining the $\alpha-$, $\beta-$ and $\gamma-$duals, characterizing some matrix classes on $\ell_p(\tilde{B})$ and geometric properties like uniform convexity, strict convexity, super reflexivity for the resulting spaces are given.

We will use the sequence y_i as the $\tilde{B}-$transform of a sequence $t=(t_j)$, i.e.,

$$y_i=(\tilde{B}t)_i=\frac{1}{B_{i+1}}\sum_{j=0}^{i}\binom{i}{j}B_j t_j,$$

for all $i\in\mathbb{N}$ whereas the relation

$$t_j=\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}\frac{B_{k+1}}{B_j}y_k, \tag{1.2}$$

holds for all $j\in\mathbb{N}$.

The main purpose of this paper is to define new paranormed space $\ell_p(\tilde{B},p)$, which is the set of all sequences whose $\tilde{B}-$transform is in the space $\ell(p)$. In other words, we generalize some conclusions, obtained in Karakas (Karakas, 2023). We examine a few topological features, including completeness, the Schauder basis, the $\alpha-$, $\beta-$ and $\gamma-$duals. On these spaces, several matrix mappings are also classified.

II. New paranormed sequence space

In this section, we give a new paranormed space by means of the Bell matrix, then, present the Schauder basis of the resulting paranormed space and prove its completeness.

Define the sequence space

$$\ell(\tilde{B},p)=\left\{t=(t_i)\in\omega:\sum_i\left|\frac{1}{B_{i+1}}\sum_{j=0}^{i}\binom{i}{j}B_j t_j\right|^{p_i}<\infty\right\}.$$

Let us begin with the completeness of this new space.

Theorem 1. *The sequence space $\ell(\tilde{B},p)$ is a complete linear metric space paranormed by*

$$\varsigma_{\tilde{B}}(t)=\left(\sum_i\left|\frac{1}{B_{i+1}}\sum_{j=0}^{i}\binom{i}{j}B_j t_j\right|^{p_i}\right)^{1/S}.$$

Proof. For $t=(t_j)$ and $y=(y_j)$ in $\ell(\tilde{B},p)$, it follows from Maddox (Maddox, 1988) that

$$\left(\sum_i\left|\frac{1}{B_{i+1}}\sum_{j=0}^{i}\binom{i}{j}B_j\left(t_j+y_j\right)\right|^{p_i}\right)^{1/S} \tag{2.1}$$

$$\leq\left(\sum_i\left|\frac{1}{B_{i+1}}\sum_{j=0}^{i}\binom{i}{j}B_jt_j\right|^{p_i}\right)^{1/S}+\left(\sum_i\left|\frac{1}{B_{i+1}}\sum_{j=0}^{i}\binom{i}{j}B_jy_j\right|^{p_i}\right)^{1/S}.$$

It can be inferred from (1.1) and (2.1) that $\ell\left(\tilde{B},p\right)$ is linear with respect to scalar multiplication and the coordinatewise addition. Also, it is obvious that $\varsigma_{\tilde{B}}\left(\theta\right)=0$ and $\varsigma_{\tilde{B}}\left(-t\right)=\varsigma_{\tilde{B}}\left(t\right)$ for all t in $\ell\left(\tilde{B},p\right)$. So, we reach the subadditivity of $\varsigma_{\tilde{B}}$ and $\varsigma_{\tilde{B}}\left(\xi t\right)\leq\max\left\{1,\left|\xi\right|\right\}\varsigma_{\tilde{B}}\left(t\right)$ for any $\xi\in\mathrm{R}$ in view of (1.1) and (2.1).

Now, say that $\left\{t^i\right\}$ is any sequence in $\ell\left(\tilde{B},p\right)$ that satisfies $\varsigma_{\tilde{B}}\left(t^i-t\right)\to 0$ and $\left(\xi_i\right)$ is any sequence in R such that $\xi_i\to\xi$. From the subadditivity of $\varsigma_{\tilde{B}}$, it can be readily written

$$\varsigma_{\tilde{B}}\left(t^i\right)\leq\varsigma_{\tilde{B}}\left(t\right)+\varsigma_{\tilde{B}}\left(t^i-t\right),$$

which yields the boundedness of $\varsigma_{\tilde{B}}\left(t^i\right)$, and the fact

$$\varsigma_{\tilde{B}}\left(\xi_i t^i-\xi t\right)=\left(\sum_i\left|\frac{1}{B_{i+1}}\sum_{j=0}^{i}\binom{i}{j}B_j\left(\xi_i t_j^i-\xi t_j\right)\right|^{p_i}\right)^{1/S}$$

$$\leq\left|\xi_i-\xi\right|\varsigma_{\tilde{B}}\left(t^i\right)+\left|\xi\right|\varsigma_{\tilde{B}}\left(t^i-t\right)\to 0,\left(\text{as} i\to\infty\right)$$

that shows the continuity of scalar multiplication. Then, $\varsigma_{\tilde{B}}$ is paranorm on $\ell\left(\tilde{B},p\right)$. In order to demonstrate the completeness of $\ell\left(\tilde{B},p\right)$, let us take any Cauchy sequence $\left\{d^n\right\}$ in $\ell\left(\tilde{B},p\right)$ such that $d^n=\left(d_1^n,d_2^n,...\right)$ for each $n\in\mathrm{N}$. For a given $\varepsilon>0$, there exists an integer $i_0\left(\varepsilon\right)\in\mathrm{N}$ such that

$$\varsigma_{\tilde{B}}\left(d^n-d^k\right)<\varepsilon \tag{2.2}$$

for all $n,k\geq i_0\left(\varepsilon\right)$. Using the definition of $\varsigma_{\tilde{B}}$, one writes that

$$\tilde{B}_i\left(d^n-d^k\right)\leq\left(\sum_i\left|\tilde{B}_i\left(d^n\right)-\tilde{B}_i\left(d^k\right)\right|^{p_i}\right)^{1/S}<\varepsilon,$$

for each $n,k\geq i_0\left(\varepsilon\right)$, this results in $\left\{\tilde{B}_i\left(d^1\right),\tilde{B}_i\left(d^2\right),...\right\}$ being a Cauchy sequence of real numbers for each fixed $i\in\mathrm{N}$. Since R is complete, we get $\tilde{B}_i\left(d^n\right)\to\tilde{B}_i\left(d\right)$, as $n\to\infty$ for every fixed $i\in\mathrm{N}$. Taking into consideration these infinitely many limits $\tilde{B}_1\left(d\right),\tilde{B}_2\left(d\right),...$, let us define the sequence $\left\{\tilde{B}_1\left(d\right),\tilde{B}_2\left(d\right),...\right\}$. It follows from (2.2) that

$$\sum_{i=1}^{m}\left|\tilde{B}_i\left(d^n\right)-\tilde{B}_i\left(d^k\right)\right|^{p_i}\leq\varsigma_{\tilde{B}}\left(d^n-d^k\right)^S<\varepsilon^S, \tag{2.3}$$

for all fixed $m\in N$ and $n,k\geq i_0(\varepsilon)$. If we take the limit $m\to\infty$ and $k\to\infty$ in (2.3), then we have $\varsigma_{\tilde{B}}\left(d^n-d\right)<\varepsilon$. If we let $\varepsilon=1$ in (2.3) and so $i\geq i_0(1)$, and if we apply the Minkowski's inequality, the we have

$$\left(\sum_{i=1}^{m}\left|\tilde{B}_i(d)\right|^{p_i}\right)^{1/S}\leq\varsigma_{\tilde{B}}\left(d^n-d\right)+\varsigma_{\tilde{B}}\left(d^n\right)\leq 1+\varsigma_{\tilde{B}}\left(d^n\right),\text{foreveryfixed}m\in N$$

from which we reach that $d\in\ell\left(\tilde{B},p\right)$. Since $\varsigma_{\tilde{B}}\left(d^n-d\right)\leq\varepsilon$ for all $n\geq i_0(\varepsilon)$, we have $d^n\to d$, as $n\to\infty$. As a result, the completeness of $\ell\left(\tilde{B},p\right)$ is established.

Theorem 2. *The sequence space* $\ell\left(\tilde{B},p\right)$ *is linearly isomorphic to* $\ell(p)$. *Here,* $0<p_i\leq S<\infty$.

Proof. For any $t\in\ell\left(\tilde{B},p\right)$, let $Q:\ell\left(\tilde{B},p\right)\to\ell(p)$ be a transformation such that $Q(t)=\left(\tilde{B}_i(t)\right)$. The linearity of Q is clear. As well, the transformation Q is injective since the fact $t=\theta$ is satisfied follows from $Q(t)=\theta$. For $y=(y_i)\in\ell(p)$, if the sequence $z=(t_j)$ is denoted for $i\in N$ by

$$t_j=\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}\frac{B_{k+1}}{B_j}y_k$$

then, we have

$$\varsigma_{\tilde{B}}(t)=\left(\sum_i\left|\frac{1}{B_{i+1}}\sum_{j=0}^{i}\binom{i}{j}B_jt_j\right|^{p_i}\right)^{1/S}$$

$$=\left(\sum_i\left|\frac{1}{B_{i+1}}\sum_{j=0}^{i}\binom{i}{j}B_j\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}\frac{B_{k+1}}{B_j}y_k\right|^{p_i}\right)^{1/S}$$

$$=\left(\sum_i\left|y_i\right|^{p_i}\right)^{1/S}=\varsigma_{\tilde{B}}(y)<\infty.$$

Hence, we find that $t\in\ell\left(\tilde{B},p\right)$. Consequently, Q is surjective and paranorm preserving. The proof is complete.

Now, we recall the definition of Schauder basis prior to establishing the Schauder basis for the matrix domain of our particular matrix. Let (H,ς) be a paranormed space. A sequence $\delta_j \in H$ is a Schauder basis for H if and only if there exists a unique sequence of scalars ε_i such that $\varsigma\left(t-\sum_{j=0}^{i}\varepsilon_j\delta_j\right)\to 0$ as $i\to\infty$. Then, we write

$$t=\sum_j \varepsilon_j\delta_j.$$

In order to provide a Schauder basis for our paranormed sequence space, we are in a position to do so.

Theorem 3. *Define a sequence* $r^{(j)}=\left(r_i^{(j)}\right)$ *in* $\ell\left(\tilde{B},p\right)$ *as*

$$r_i^{(j)}=\begin{cases}(-1)^{i-j}\binom{i}{j}\dfrac{B_{j+1}}{B_i}, & \text{if} 0\le j\le i;\\ 0, & \text{if} j>i;\end{cases}$$

where $i\in\mathrm{N}$ is fixed. Then, the sequence $\left\{r^{(0)},r^{(1)},\ldots\right\}$ is a Schauder basis for the space $\ell\left(\tilde{B},p\right)$ and any t in $\ell\left(\tilde{B},p\right)$ is uniquely determined as

$$t=\sum_j y_j r^{(j)}, \tag{2.4}$$

where $y_j=\left(\tilde{B}t\right)_j$.

Proof. Firstly, it is clear that

$$\tilde{B}r^{(i)}=e^{(i)}\in\ell(p) \tag{2.5}$$

for $0<p_i\le P<\infty$. Let $t\in\ell(\tilde{B},p)$ and denote

$$t^{[v]}=\sum_{i=0}^{v}(\tilde{B}t)_i r^{(i)} \tag{2.6}$$

for every non-negative integer v.

Equations (2.5) and (2.6) are now used to determine that

$$\begin{aligned}\tilde{B}t^{[v]}&=\sum_{i=0}^{v}(\tilde{B}t)_i\tilde{B}r^{(i)}\\ &=(\tilde{B}t)_i e^{(i)},\end{aligned}$$

and

$$\left(\tilde{B}(t-t^{[v]})\right)_m=\begin{cases}0 & (0\le m\le v),\\ (\tilde{B}t)_m & (m>v).\end{cases}$$

Now, there exists an integer v_0 such that for a given $\varepsilon>0$

$$\left(\sum_{m\geq v}\left|(\tilde{B}t)_m\right|^{p_m}\right)^{1/S}<\frac{\varepsilon}{2}$$

for all $v\geq v_0$. As a result, we can see for all $v\geq v_0$ that

$$\varsigma_{\tilde{B}}\left(t-t^{[v]}\right)=\left(\sum_{m\geq v}\left|(\tilde{B}t)_m\right|^{p_m}\right)^{1/S}$$

$$\leq\left(\sum_{m\geq v_0}\left|(\tilde{B}t)_m\right|^{p_m}\right)^{1/S}<\frac{\varepsilon}{2}<\varepsilon,$$

which yields (2.4). For the purpose of the uniqueness of the representation (2.4), let

$$t=\sum_i y' r^{(i)}$$

be another representation of t. Then, we have

$$(\tilde{B}t)_m=\sum_i y_i'\left(\tilde{B}r^{(i)}\right)_m=\sum_i y_i' e_m^{(i)}=y_m',\quad (m\in \mathrm{N}_0).$$

Thus, the representation (2.4) is unique. This completes the proof.

III. The $\alpha-$, $\beta-$, and $\gamma-$duals

This section deals with $\alpha-$dual, $\beta-$dual and $\gamma-$dual of $\ell\left(\tilde{B},p\right)$. We give some definitions and lemmas before our results.

The following set $S\left(H,G\right)$ is the multiplier space of H and G defined by

$$S\left(H,G\right)=\left\{z\in\omega : zx\in G\text{forall}x\in H\right\}.$$

The definitions of the $\alpha-$, $\beta-$ and $\gamma-$duals of a sequence space H using this notation are as follows

$$H^{\alpha}=S\left(H,\ell_1\right),\, H^{\beta}=S\left(H,cs\right)\text{and}H^{\gamma}=S\left(H,bs\right).$$

In this context, cs and bs represent, respectively, the spaces of sequences with convergent and bounded series.

Lemma 4. (Grosseerdmann, 1993)

1) Let $1<p_n\leq P<\infty$ for all $n\in \mathrm{N}$. Then, $U=\left(u_{nk}\right)\in\left(\ell(p),\ell_1\right)$ if and only if

$$\sup_{N\in ô}\sum_k\left|\sum_{n\in N}u_{nk}K^{-1}\right|^{p_n'}<\infty$$

for some integer $K>1$. ô stands for the family of all finite subsets of N in this context.

2) Let $0 < p_n \leq 1$ for all $n \in \mathbb{N}$. Then, $U = (u_{nk}) \in (\ell(p), \ell_1)$ if and only if

$$\sup_{N \in \mathcal{F}} \sup_k \left| \sum_{n \in N} u_{nk} \right|^{p_n} < \infty.$$

Lemma 5. (Lascarides and Maddox 1970)

1) Let $1 < p_n \leq P < \infty$ for all $n \in \mathbb{N}$. Then, $U = (u_{nk}) \in (\ell(p), \ell_\infty)$ if and only if

$$\sup_n \sum_k \left| u_{nk} K^{-1} \right|^{p_n'} < \infty \tag{3.1}$$

for some integer $K > 1$.

2) Let $0 < p_n \leq 1$ for all $n \in \mathbb{N}$. Then, $U = (u_{nk}) \in (\ell(p), \ell_\infty)$ if and only if

$$\sup_{n,k \in \mathbb{N}} \left| u_{nk} \right|^{p_n} < \infty. \tag{3.2}$$

Lemma 6. (Lascarides and Maddox 1970) Let $0 < p_n \leq P < \infty$ for all $n \in \mathbb{N}$. Then, $U = (u_{nk}) \in (\ell(p), c)$ if and only if (3.1) and (3.2) hold and

$$\lim_n u_{nk} = \eta_k, \ (k \in \mathbb{N}) \tag{3.3}$$

for some η_k.

Theorem 7. Let $M \in \mathbb{N}$ and $I > 1$. We define the following sets:

$$\tau_1 := \left\{ u = (u_n) \in \omega : \sup_{M \in \mathbb{N}} \sup_{n \in \mathbb{N}} \left| \sum_{m \in M} (-1)^{m-n} \binom{m}{n} \frac{B_{n+1}}{B_m} u_m \right|^{p_n} < \infty \right\},$$

$$\tau_2 := \bigcup_{I > 1} \left\{ u = (u_n) \in \omega : \sup_{M \in \mathbb{N}} \sum_n \left| \sum_{m \in M} (-1)^{m-n} \binom{m}{n} \frac{B_{n+1}}{B_m} u_m I^{-1} \right|^{p_n'} < \infty \right\}.$$

Then $\left[\ell(\tilde{B}, p) \right]^\alpha = \begin{cases} \tau_1 & , \quad 0 < p_n \leq 1, \\ \tau_2 & , \quad 1 < p_n \leq P < \infty. \end{cases}$

Proof. We notice that the equality

$$u_m z_m = \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} \frac{B_{n+1}}{B_m} u_m y_n = (Ry)_m$$

holds in the presence of (1.2) for $u = (u_m) \in \omega$, where $R = (r_{mn})$ is a triangle defined by

$$r_{mn} = \begin{cases} (-1)^{m-n}\binom{m}{n}\dfrac{B_{n+1}}{B_m}u_m & , \quad 0 \le n \le m, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Thus $uz = (u_m z_m) \in \ell_1$ whenever $z \in \ell(\tilde{B}, p)$ if and only if $Ry \in \ell_1$ whenever $y \in \ell(p)$. This implies that $u = (u_m) \in \left[\ell(\tilde{B}, p)\right]^{\alpha}$ if and only if $R \in (\ell(p), \ell_1)$. Thus by using Lemma 4, we realise that

$$\exists I > 1 å \sup_{M \in \mathrm{N}} \sum_n \left| \sum_{m \in M} (-1)^{m-n}\binom{m}{n}\frac{B_{n+1}}{B_m}u_m I^{-1}\right|^{p_n'} < \infty, 1 < p_n \le P < \infty,$$

$$\sup_{M \in \mathrm{N}} \sup_{n \in \mathrm{N}} \left| \sum_{m \in M} (-1)^{m-n}\binom{m}{n}\frac{B_{n+1}}{B_m}u_m\right|^{p_n} < \infty, 0 < p_n \le 1.$$

This concludes that

$$\left[\ell(\tilde{B}, p)\right]^{\alpha} = \begin{cases} \tau_1 & , \quad 0 < p_n \le 1, \\ \tau_2 & , \quad 1 < p_n \le P < \infty. \end{cases}$$

Theorem 8. Consider the sets τ_3, τ_4, τ_5 as follows:

$$\tau_3 := \left\{ u = (u_n) \in \omega : \sup_{m,n \in \mathrm{N}} \left| \sum_{j=n}^{m} (-1)^{j-n}\binom{j}{n}\frac{B_{n+1}}{B_j}u_j \right|^{p_n} < \infty \right\},$$

$$\tau_4 := \left\{ u = (u_n) \in \omega : \sum_{j=n}^{\infty} (-1)^{j-n}\binom{j}{n}\frac{B_{n+1}}{B_j}u_j \text{isconvergent} \right\},$$

$$\tau_5 := \bigcup_{I>1} \left\{ u = (u_n) \in \omega : \sup_m \sum_n \left| \sum_{j=n}^{m} (-1)^{j-n}\binom{j}{n}\frac{B_{n+1}}{B_j}u_j I^{-1} \right|^{p_n'} \right\}.$$

Then

$$\left[\ell(\tilde{B}, p)\right]^{\beta} = \begin{cases} \tau_3 \cap \tau_4 & , \quad 0 < p_n \le 1, \\ \tau_4 \cap \tau_5 & , \quad 1 < p_n \le P < \infty. \end{cases}$$

Proof. Take any $u = (u_n) \in \omega$. Since $y = (y_n)$ is the $\tilde{B}$-transform of the sequence $z = (z_n)$, we write

$$\sum_{j=0}^{m} u_j z_j = \sum_{j=0}^{m} (\sum_{n=0}^{j} (-1)^{j-n}\binom{j}{n}\frac{B_{n+1}}{B_j}u_j) y_n = E_m(y) \tag{3.4}$$

where the matrix $E = (e_{mn})$ is defined by

$$e_{mn}=\begin{cases}\sum_{j=n}^{m}(-1)^{j-n}\binom{j}{n}\frac{B_{n+1}}{B_j}u_j & , \quad 1\le n\le m,\\ 0 & , \quad \text{otherwise.}\end{cases}$$

for all $m,n\in\mathbb{N}$. Therefore, using Lemma 6 with (3.4), we have that $uz=(u_jz_j)\in cs$ whenever $z=(z_j)\in\ell(\tilde{B},p)$ if and only if $Ey\in c$ whenever $y=y_n\in\ell(p)$ which gives us $u=(u_j)\in\left[\ell(\tilde{B},p)\right]^{\beta}$ if and only if $E\in(\ell(p),c)$. Thus, it concludes from (3.1) and (3.3) that $\left[\ell(\tilde{B},p)\right]^{\beta}=\tau_4\cap\tau_5$. The proof of the case $0<p_n\le 1$ can be established in a similar way.

Theorem 9. $\left[\ell(\tilde{B},p)\right]^{\gamma}=\begin{cases}\tau_3 & , \quad 0<p_n\le 1,\\ \tau_5 & , \quad 1<p_n\le P<\infty,\end{cases}$

Proof. From Lemma 5 and equality (3.4), we see that $uz=(u_jz_j)\in bs$ whenever $z=(z_j)\in\ell(\tilde{B},p)$ if and only if $Ey\in\ell_\infty$ whenever $y=y_n\in\ell(p)$ where the matrix $E=(e_{mn})$ is defined as in the proof of previous theorem. So, we obtain the expected result from (3.1) and (3.2).

IV. Matrix transformations

In the present section, our aim is to characterize some matrix classes on the space $\ell(\tilde{B},p)$. The following theorem includes the exact conditions for $0<p_n\le P<\infty$ since the cases $1<p_n\le P<\infty$ and $0<p_n\le 1$ are combined. So, we'll only prove for $1<p_n\le P<\infty$ since the other case can be obtained using similar method.

Theorem 10.

1. Let $0<p_n\le 1$ for all $n\in\mathbb{N}$. Then, $V=(v_{mn})\in(\ell(\tilde{B},p),\ell_\infty)$ if and only if

$$\sup_{m,n\in\mathbb{N}}\left|\sum_{j=n}^{\infty}(-1)^{j-n}\binom{j}{n}\frac{B_{n+1}}{B_j}v_{mj}\right|^{p_n}<\infty \tag{4.1}$$

$$\sum_{j=n}^{\infty}(-1)^{j-n}\binom{j}{n}\frac{B_{n+1}}{B_j}v_{mj}<\infty. \tag{4.2}$$

2. Let $1<p_n\le P<\infty$ for all $n\in\mathbb{N}$. Then, $V=(v_{mn})\in(\ell(\tilde{B},p),\ell_\infty)$ if and only if (4.2) holds and there is an integer $I>1$ such that

$$\sup_{m\in\mathbb{N}}\sum_n\left|\sum_{j=n}^{\infty}(-1)^{j-n}\binom{j}{n}\frac{B_{n+1}}{B_j}v_{mj}I^{-1}\right|^{p_n'}<\infty. \tag{4.3}$$

Proof. Let $1 < p_n \le P < \infty$ for all $n \in \mathrm{N}$ and $V \in (\ell(\tilde{B}, p), \ell_\infty)$. Then for every $z \in \ell(\tilde{B}, p)$, Vz exists and so, $V_m \in \left[\ell(\tilde{B}, p)\right]^\beta$ for every $m \in \mathrm{N}$. Thus, the necessary conditions for (4.2) and (4.3) are obvious. For sufficient conditions, let us assume that $z \in \ell(\tilde{B}, p)$ and (4.2), (4.3) hold. Then, Vz, that is, V-transform of z, exists since $V_m \in \left[\ell(\tilde{B}, p)\right]^\beta$ for every $m \in \mathrm{N}$. Thus, we have by (1.2) that

$$\sum_{j=0}^{k} v_{mj} z_j = \sum_{j=0}^{k} \sum_{n=0}^{j} (-1)^{j-n} \binom{j}{n} \frac{B_{n+1}}{B_j} y_n v_{mj}$$

$$= \sum_{n=0}^{k} \sum_{j=n}^{k} (-1)^{j-n} \binom{j}{n} \frac{B_{n+1}}{B_j} v_{mj} y_n \tag{4.4}$$

for all $m, k \in \mathrm{N}$. In view of the hypothesis, we obtain from (4.4) that

$$\sum_j v_{mj} z_j = \sum_n \sum_{j=n}^{\infty} (-1)^{j-n} \binom{j}{n} \frac{B_{n+1}}{B_j} v_{mj} y_n \tag{4.5}$$

for all $m \in \mathrm{N}$ as $k \to \infty$. If we consider (4.5) and the following inequality for any $c, d \in \mathrm{C}$ and $I > 0$ with $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$,

$$|cd| \le I(|cI^{-1}|^{p'} + |d|^p)$$

we have

$$\sup_{m \in \mathrm{N}} \left| \sum_j v_{mj} z_j \right| = \sup_{m \in \mathrm{N}} \left| \sum_n \sum_{j=n}^{\infty} (-1)^{j-n} \binom{j}{n} \frac{B_{n+1}}{B_j} v_{mj} y_n \right|$$

$$\le \sup_{m \in \mathrm{N}} \sum_n \left| \sum_{j=n}^{\infty} (-1)^{j-n} \binom{j}{n} \frac{B_{n+1}}{B_j} v_{mj} y_n \right|$$

$$\le \sup_{m \in \mathrm{N}} \sum_n I\left(\left| \sum_{j=n}^{\infty} (-1)^{j-n} \binom{j}{n} \frac{B_{n+1}}{B_j} v_{mj} I^{-1} \right|^{p_n'} + |y_n|^{p_n} \right)$$

$$= I\left(\sup_{m \in \mathrm{N}} \sum_n \left| \sum_{j=n}^{\infty} (-1)^{j-n} \binom{j}{n} \frac{B_{n+1}}{B_j} v_{mj} I^{-1} \right|^{p_n'} + \sup_{m \in \mathrm{N}} \sum_n |y_n|^{p_n} \right)$$

$$< \infty.$$

This yields that $Vz \in \ell_\infty$.

Theorem 11.

1. Let $0 < p_n \le 1$ for all $n \in \mathbb{N}$. Then, $V = (v_{mn}) \in (\ell(\tilde{B}, p), c)$ if and only if the conditions (4.1), (4.2) hold and

$$\lim_{m\to\infty} \sum_{j=n}^{\infty} (-1)^{j-n} \binom{j}{n} \frac{B_{n+1}}{B_j} v_{mj} = \gamma_n \text{ for all } n \in \mathbb{N}. \tag{4.6}$$

2. Let $1 < p_n \le P < \infty$ for all $n \in \mathbb{N}$. Then, $V = (v_{mn}) \in (\ell(\tilde{B}, p), c)$ if and only if the conditions (4.2), (4.3), (4.6) hold.

Proof. Let $1 < p_n \le P < \infty$ for all $n \in \mathbb{N}$ and $V \in (\ell(\tilde{B}, p), c)$. Then, the necessity conditions for (4.2) and (4.3) are obviously seen from previous theorem and the inclusion $c \subset \ell_\infty$.

Let's now consider the sequence $s^{(n)}$ that was specified in the proof of Theorem 3 to show that the necessity condition for (4.6). Since Vz, i.e. the V-transform of every $z \in \ell(\tilde{B}, p)$, exists and is in c, we obtain

$$Vs^{(n)} = \left(\sum_{j=0}^{\infty} v_{ij} s_j^{(n)}\right)_{i=0}^{\infty} = \left(\sum_{j=n}^{\infty} (-1)^{j-n} \binom{j}{n} \frac{B_{n+1}}{B_j} v_{ij}\right)_{i=0}^{\infty} \in c$$

for every fixed $n \in \mathbb{N}$. This yields the necessity of (4.6).

For sufficiency, let us assume that $z \in \ell(\tilde{B}, p)$ and (4.2), (4.3), (4.6) hold. So, Vz exists.

Hence, for all $k, m \in \mathbb{N}$, the inequality

$$\sum_{n=0}^{k} \left| \sum_{j=n}^{m} (-1)^{j-n} \binom{j}{n} \frac{B_{n+1}}{B_j} v_{mj} I^{-1} \right|^{p_n'}$$

$$\le \sup_{m\in\mathbb{N}} \sum_{n} \left| \sum_{j=n}^{\infty} (-1)^{j-n} \binom{j}{n} \frac{B_{n+1}}{B_j} v_{mj} I^{-1} \right|^{p_n'} < \infty.$$

This yields the following fact with (4.3) and (4.6) as $k, m \to \infty$

$$\lim_{k,m\to\infty} \sum_{n=0}^{k} \left| \sum_{j=n}^{m} (-1)^{j-n} \binom{j}{n} \frac{B_{n+1}}{B_j} v_{mj} I^{-1} \right|^{p_n'}$$

$$\le \sup_{m\in\mathbb{N}} \sum_{n} \left| \sum_{j=n}^{\infty} (-1)^{j-n} \binom{j}{n} \frac{B_{n+1}}{B_j} v_{mj} I^{-1} \right|^{p_n'}.$$

Therefore, we mean that $\sum_n \left|\gamma_n I^{-1}\right|^{p_n'}$ and $(\gamma_n) \in \left[\ell(\tilde{B}, p)\right]^{\beta}$ which indicates that the series $\sum_n \gamma_n z_n$ converges for all $z \in \ell(\tilde{B}, p)$.

Lastly, take into consideration the following equality derived from (4.5) with $(v_{mj} - \gamma_j)$ in place of v_{mj}

$$\sum_j (v_{mj} - \gamma_j) z_j = \sum_n \sum_{j=n}^{\infty} (-1)^{j-n} \binom{j}{n} \frac{B_{n+1}}{B_j} (v_{mj} - \gamma_j) y_n$$
$$= \sum_n \rho_{mn} y_n$$

where $\rho_{mn} = \sum_{j=n}^{\infty} (-1)^{j-n} \binom{j}{n} \frac{B_{n+1}}{B_j} (v_{mj} - \gamma_j)$ for all $m, n \in \mathrm{N}$. Thus, $\rho_{mn} \to 0$ as $m \to \infty$ for all $n \in \mathrm{N}$ by Lemma 6. Hence, we have $\sum_n (v_{mn} - \gamma_n) z_n \to 0$ as $m \to \infty$. This concludes that $Vz \in c$ whenever $z \in \ell(\tilde{B}, p)$.

Corollary 12.

1. Let $0 < p_n \le 1$ for all $n \in \mathrm{N}$. Then, $v = (v_{mn}) \in (\ell(\tilde{B}, p), c_0)$ if and only if the conditions (4.1), (4.2) and (4.6) with $\gamma_n = 0$ for all $n \in \mathrm{N}$ hold.

2. Let $1 < p_n \le P < \infty$ for all $n \in \mathrm{N}$. Then, $v = (v_{mn}) \in (\ell(\tilde{B}, p), c_0)$ if and only if the conditions (4.2), (4.3) and (4.6) with $\gamma_n = 0$ for all $n \in \mathrm{N}$ hold.

Availability of Data and Materials: No data were used to support the findings of the study.

Conflicts of Interest: The authors declare that they have no conflict of interests.

Funding: There was no funding support for this study.

Authors' Contributions: All authors contributed equally to this work. All the authors have read and approved the final version manuscript.

References:

Alp, P.Z. (2020). A new paranormed sequence space defined by Catalan conservative matrix. Mathematical Methods in Applied Sciences, 44 (9), 7651–7658.

Altay, B., Başar, F. (2006). Some paranormed sequence spaces of non-absolute type derived by weighted mean. Journal of Mathematical Analysis and Applied Mathematics, 319 (2), 494–508.

Aydın, C., Başar, F. (2004). Some new paranormed sequence spaces. Information Sciences, 160 (1-4), 27–40.

Candan, M. (2012). Domain of the double sequential band matrix in the classical sequence spaces. Journal of Inequalities and Applications, 2012, 281.

Candan, M., Güneş, A. (2015). Paranormed sequence space of non-absolute type founded using generalized difference matrix. Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, 85 (2), 269–276.

Choudhary, B., Mishra, S.K. (1993). On Köthe-Toeplitz duals of certain sequence spaces and their matrix transformations. Indian Journal of Pure and Applied Mathematics, 24 (5), 291–301.

Dağlı, M.C. (2023). On the paranormed sequence space arising from Catalan–Motzkin matrix. Advances in Operator Theory, 8 (2), 33.

Dağlı, M.C., Yaying, T. (2023). Some new paranormed sequence spaces derived by regular Tribonacci matrix. The Journal of Analysis, 31, 109–127.

Et, M., Çolak, R. (1995). On some generalized difference sequence spaces. Soochow Journal of Mathematics, 21 (4), 377–386.

Grosseerdmann, K.G. (1993). Matrix transformations between the sequence spaces of Maddox. Journal of Mathematical Analysis and Applications, 180 (1), 223–238.

İlkhan, M., Kara, E.E., Usta, F. (2020). Compact operators on the Jordan totient sequence spaces. Mathematical Methods in Applied Sciences, 44 (9), 7666–7675.

İlkhan, M., Simsek, N., Kara, E.E. (2020). A new regular infinite matrix defined by Jordan totient function and its matrix domain in ℓ_p. Mathematical Methods in Applied Sciences, 44 (9), 7622–7633.

İlkhan, M., Demiriz, S., Kara, E.E. (2019). A new paranormed sequence space defined by Euler totient matrix. Karaelmas Science and Engineering Journal, 9 (2), 277–282.

Kara E.E., Demiriz, S. (2015). Some new paranormed difference sequence spaces derived by Fibonacci numbers. Miskolc Mathematical Notes, 16 (2), 907–923.

Karakas, M. (2023). On the sequence spaces involving bell numbers. Linear and Multilinear Algebra, 71 (14), 2298-2309.

Karakaya, V., Noman, A.K., Polat, H. (2011). On paranormed $\lambda-$sequence spaces of non-absolute type. Mathematical and Computer Modelling, 54, 1473–1480.

Karakaya, V., Savaş, E., Polat, H. (2013). Some paranormed Euler sequence spaces of difference sequences of order m. Mathematica Slovaca, 63, 849–862.

Karakaya, V., Şimşek, N. (2012). On some properties of new paranormed sequence space of nonabsolute type. Abstract and Applied Analysis, 2012, 921613.

Kirişçi, M., Başar, F. (2010). Some new sequence spaces derived by the domain of generalized difference matrix. Computers & Mathematics with Applications, 60, 1299-1309.

Lascarides, C.G., Maddox, I.J. (1970). Matrix transformations between some classes of sequences. Mathematical Proceedings of the Cambridge Philosophical Society, 68, 99-104.

Maddox, I.J. (1967). Spaces of strongly summable sequences. The Quarterly Journal of Mathematics, 18 (2), 345–355.

Maddox, I.J. (1968). Paranormed sequence spaces generated by infinite matrices. Mathematical Proceedings of the Cambridge Philosophical Society, 64, 335–340.

Maddox, I.J. (1988). Elements of Functional Analysis. The University Press, Cambridge.

Malkowsky, E. (1997). Recent results in the theory of matrix transformations in sequence spaces. Matematički Vesnik, 49, 187–196.

Malkowsky, E., Özger, F., Velickovic, V. (2017). Some mixed paranorm spaces. Filomat, 31, 1079–1098.

Malkowsky, E., Özger, F., Velickovic, V. (2017). Matrix transformations on mixed paranorm spaces. Filomat, 31, 2957–2966.

Malkowsky, E., Savas, E. (2004). Matrix tansformations between sequence spaces of generalized weighted mean. Applied Mathematics and Computation, 147, 333-345.

Nakano, H. (1951). Modulared sequence spaces, Proceedings of the Japan Academy, Series A, Mathematical Sciences, 27 (2), 508-512.

Simons, S. (1965). The sequence spaces $\ell(p_v)$ and $m(p_v)$,. Proceedings of the London Mathematica Society, 15 (3), 422–436.

Yaying, T. (2022). Paranormed Riesz difference sequence spaces of fractional order. Kragujevac Journal of Mathematics, 46 (2), 175–191.