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Research Article

On Gradual Borel Summability Method of Rough Convergence of Triple Sequences of Beta Stancu Operators

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Abstract: We define the concept of rough limit set of a triple sequence space of beta Stancu operators of Borel summability of gradual real numbers and obtain the relation between the set of rough limit and the extreme limit points of a triple sequence space of beta Stancu operators of Borel summability method of gradual real numbers. Finally, we investigate some properties of the rough limit set of beta Stancu operators under which Borel summable sequence of gradual real numbers are convergent. Also, we give the results for Borel summability method of series of gradual real numbers.

Keywords: Gradual real number, Triple sequences, Rough convergence, Closed and convex, Cluster points and rough limit points, Sequences of fuzzy interval, Beta Stancu operators, Borel summability method.

MSC Subject classification: 40F05, 40J05, 40G05

I. Introduction

The term "fuzzy numbers" are often applied instead of "fuzzy intervals", especially if the core of fuzzy interval is a point (like; triangular fuzzy number). But such fuzzy numbers also generalize intervals not numbers. Also fuzzy arithmetics in herit algebraic properties of interval arithmetic, not of numbers. Hence the name "fuzzy number", used by many authors is debatable. To avoid this confusion, the authors introduce a new concept in fuzzy set theory as "gradual real numbers". A gradual number in general cannot be considered as a fuzzy set of real numbers because the mapping from the unit interval to the real line is not necssarily one to one. However gradual real numbers are equipped with the same algebraic structures as real numbers.

In this paper, we introduce and study the set of rough gradual real numbers along with triple sequence spaces of

Borel summability method and some basic topological properties of it.

Let X be a real vector space and $\|.\|_G$ be a mapping from $X \to G^*(\mathbb{R}^3)$. Consider A be a assignment function language. A triple sequence $x \in (S_{mnk})$ and $\lambda \in (0,1]$ is said to be triple gradual Borel summable to S if the series $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^{m+n+k}}{(m+n+k)!} A_{\|.\|_G,S_{mnk}}(\lambda)$ converges for all $x \in \mathbb{R}$ and

$$e^{-x} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^{m+n+k}}{(m+n+k)!} A_{\|.\|_G,S_{mnk}}(\lambda) \to S, x \to \infty.$$

In this we define triple rough Borel summability method for sequences and series of gradual real numbers.

Definition 1.1. Let X be a real vector space and $\|.\|_G$ be a mapping from $X \to G^*(\mathbb{R}^3)$. A be a assignment function language . Consider (u_{mnk}) be a triple sequence of gradual real numbers and $\lambda \in (0,1]$. Then the expression $\sum \sum \sum A_{\|.\|_G,u_{mnk}}(\lambda)$ is called a series of triple gradual real numbers. Denote $S_{rst} = \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} A_{\|.\|_G,u_{mnk}}(\lambda)$ for all $r, s, t \in \mathbb{N}$. If the sequence (S_{rst}) converges to a gradual real number u, then we say that the series $\sum \sum \sum A_{\|.\|_G,u_{mnk}}(\lambda)$ of gradual real numbers converges to $A_{\|.\|_G,u}(\lambda)$ and write $\sum \sum \sum A_{\|.\|_G,u_{mnk}}(\lambda) = A_{\|.\|_G,u}(\lambda)$ which implies as $r, s, t \to \infty$ that $\sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} A_{\|.\|_G,u_{mnk}^-}(\lambda) \to A_{\|.\|_G,u^-}(\lambda)$ and $\sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} A_{\|.\|_G,u_{mnk}^+}(\lambda) \to A_{\|.\|_G,u^+}(\lambda)$ uniformly in $\lambda \in (0,1]$.

Conversely, if the gradual real numbers

$$A_{\|.\|_G,u_{mnk}}(\lambda) = \left\{\left(A_{\|.\|_G,u_{mnk}^-}(\lambda), A_{\|.\|_G,u_{mnk}^+}(\lambda)\right) : \lambda \in (0,1]\right\},$$

$\sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} A_{\|.\|_G,u_{mnk}^-}(\lambda) \to A_{\|.\|_G,u^-}(\lambda)$ and $\sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} A_{\|.\|_G,u_{mnk}^+}(\lambda) \to A_{\|.\|_G,u^+}(\lambda)$ converge uniformly in λ, then $A_{\|.\|_G,u}(\lambda) = \left\{\left(A_{\|.\|_G,u^-}(\lambda), A_{\|.\|_G,u^+}(\lambda)\right) : \lambda \in (0,1]\right\}$ defines a gradual real numbers such that $u = \sum \sum \sum A_{\|.\|_G,u_{mnk}}(\lambda)$.

We say other wise the series of gradual real numbers diverges. Additionally, if the triple sequence gradual real numbers (S_{rst}) is bounded then we say that the series $\sum \sum \sum A_{\|.\|_G,u_{mnk}}(\lambda)$ of gradual real numbers is bounded. We denote the set of all bounded series of gradul real numbers by $bs(F)$.

Definition 1.2. A triple sequence (u_{mnk}) and $\lambda \in (0,1]$ of gradual real numbers is said to be Borel summable to $\zeta \in E^{`}$ if the series

$$f(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^{m+n+k}}{(m+n+k)!} A_{\|.\|_G,u_{mnk}}(\lambda)$$

converges for $x \in (0, \infty)$ and $lim_{x \to \infty} e^{-x} f(x) = \zeta$.

The idea of rough convergence was first introduced by Phu (Phu, 2001, 2002, 2003) in finite dimensional normed spaces. He showed that the set LIM_x^r is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree r.

Aytar (Aytar, 2008a) studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar (Aytar, 2008b) studied that the $r-$limit set of the sequence is equal to intersection of these sets and that $r-$core of the sequence is equal to the union of these sets. Dündar and Cakan (Dündar and Cakan, 2014) investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence The notion of $I-$convergence of a triple sequence spaces which is based on the structure of the ideal I of subsets of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

Let K be a subset of the set of triple positive integers $\mathbb{N}^3$, and let us denote the set $\{(m,n,k) \in K: m \leq u, n \leq v, k \leq w\}$ by K_{uvw}. Then the natural density of K is given by $\delta(K) = lim_{u,v,w\to\infty} \frac{|K_{uvw}|}{uvw}$, where $|K_{uvw}|$ denotes the number of elements in K_{uvw}. Clearly, a finite subset has natural density zero, and we have $\delta(K^c) = 1 - \delta(K)$ where $K^c = \mathbb{N} \backslash K$ is the complement of K. If $K_1 \subseteq K_2$, then $\delta(K_1) \leq \delta(K_2)$.

Throughout the paper, $\mathbb{R}^3$ denotes the real three dimensional space with metric (X,d). Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}, m,n,k \in \mathbb{N}$.

A triple sequence spaces $x = (x_{mnk})$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st - lim \quad x = 0$, provided that the set

$$\{(m,n,k) \in \mathbb{N}^3: |x_{mnk}| \geq \epsilon\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence spaces x.

If a triple sequence spaces is statistically convergent, then for every $\epsilon > 0$, infinitely many terms of the sequence may remain outside the $\epsilon-$neighbour hood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero. This is an important property that distinguishes statistical convergence from ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a triple sequence spaces $x = (x_{mnk})$ satisfies some property P for all m,n,k except a set of natural density zero, then,we say that the triple sequence spaces x satisfies P for "almost all (m,n,k)" and we abbreviate this by "a.a. (m,n,k)".

Let $\left(x_{m_i n_j k_\ell}\right)$ be a sub-sequence of $x = (x_{mnk})$. If the natural density of the set $K = \{m_i n_j k_\ell \in \mathbb{N}: (i,j,\ell) \in$

$\mathbb{N}^3\}$ is different from zero, then $\left(x_{m_i n_j k_\ell}\right)$ is called a non thin sub sequence of a triple sequence spaces x.

$c \in \mathbb{R}$ is called a statistical cluster point of a triple sequence spaces $x = (x_{mnk})$ provided that the natural density of the set

$$\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk} - c| < \epsilon\}$$

is different from zero for every $\epsilon > 0$. We denote the set of all statistical cluster points of the sub sequence x by Γ_x.

A triple sequence spaces $x = (x_{mnk})$ is said to be statistically analytic if there exists a positive number M such that

$$\delta\left(\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk}|^{1/m+n+k} \geq M\}\right) = 0.$$

First applied the concept of (p, q)-calculus in approximation theory and introduced the (p, q)-analogue of Bernstein operators. Later, based on (p, q)-integers, some approximation results for Bernstein-Stancu operators, Bernstein-Kantorovich operators, (p, q)-Lorentz operators, Bleimann-Butzer and Hahn operators and Bernstein-Shurer operators etc.

In computer-aided geometric design and applied these Bernstein basis for construction of (p, q)-Bezier curves and surfaces based on (p, q)-integers which is further generalization of q-Bezier curves and surfaces.

Motivated by the above mentioned work on (p, q)-approximation and its application, in this paper we study statistical approximation properties of Bernstein Stancu beta operators based on (p, q)-integers.

Now we recall some basic definitions about (p, q)-integers. For any $u, v, w \in \mathbb{N}$, the (p, q)-integer $[uvw]_{p,q}$ is defined by

$$[0]_{p,q} := 0 \text{ and } [uvw]_{p,q} = \frac{p^{uvw} - q^{uvw}}{p - q} \text{ if } u, v, w \geq 1,$$

where $0 < q < p \leq 1$. The (p, q)-factorial is defined by

$$[0]_{p,q}! := 1 \text{ and } [uvw]!_{p,q} = [1]!_{p,q}\, [6]!_{p,q}\, [uvw]!_{p,q} \text{ if } u, v, w \geq 1.$$

Also the (p, q)-binomial coefficient is defined by

$$\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q} = \frac{[uvw]!_{p,q}}{[mnk]!_{p,q}\, [(u-m) + (v-n) + (w-k)]!_{p,q}}$$

for all $u, v, w, m, n, k \in \mathbb{N}$ with $u \geq m, v \geq n, w \geq k$.

The formula for (p, q) −binomial expansion is as follows:

$$(ax + by)^{uvw}_{p,q} = \sum\nolimits_{m=0}^{u} \sum\nolimits_{n=0}^{v} \sum\nolimits_{k=0}^{w} p^{\frac{(u-m)(u-m-1)+(v-n)(v-n-1)+(w-k)(w-k-1)}{6}} q^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}}$$
$$\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q} a^{(u-m)+(v-n)+(w-k)} b^{m+n+k} x^{(u-m)+(v-n)+(w-k)} y^{m+n+k},$$

$$(x + y)^{uvw}_{p,q} = (x + y)(px + qy)(p^6 x + q^6 y) \cdots \left(p^{(u-1)+(v-1)+(w-1)} x + q^{(u-1)+(v-1)+(w-1)} y\right),$$

$$(1-x)_{p,q}^{uvw} = (1-x)(p-qx)(p^6-q^6x)\cdots\left(p^{(u-1)+(v-1)+(w-1)} - q^{(u-1)+(v-1)+(w-1)}x\right), \text{ and}$$

$$(x)_{p,q}^{mnk} = x(px)(p^6x)\cdots\left(p^{(u-1)+(v-1)+(w-1)}x\right) = p^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}}.$$

The Bernstein operator of order (r,s,t) is given by

$$B_{rst}(f,x) = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k} x^{m+n+k}(1-x)^{(m-r)+(n-s)+(k-t)}$$

where f is a continuous (real or complex valued) function defined on $[0,1]$.

The (p,q)-Bernstein operators are defined as follows:

$$B_{rst,p,q}(f,x) = \frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{6}}}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k} p^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}} x^{m+n+k}$$

$$\prod_{u=0}^{(r-m-1)+(s-n-1)+(t-k-1)} (p^u - q^u x) f\left(\frac{[mnk]_{p,q}}{p^{(m-r)+(n-s)+(k-t)}[rst]_{p,q}}\right), x \in [0,1] \quad (1.1)$$

Also, we have

$$(1-x)_{p,q}^{rst} = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}(-1)^{m+n+k} p^{\frac{(r-m)(r-m-1)+(s-n)(s-n-1)+(t-k)(t-k-1)}{6}} q^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}}$$

$$\binom{r}{m}\binom{s}{n}\binom{t}{k} x^{m+n+k}.$$

(p,q)-Bernstein Stancu operators are defined as follows:

$$S_{rst,p,q}(f,x) = \frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{6}}}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k} p^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}} x^{m+n+k}$$

$$\prod_{u=0}^{(r-m-1)+(s-n-1)+(t-k-1)} (p^u - q^u x) f\left(\frac{p^{(r-m)+(s-n)+(t-k)}[mnk]_{p,q}+\eta}{[rst]_{p,q}+\mu}\right), x \in [0,1] \quad (1.2)$$

Note that for $\eta = \mu = 0$, (p,q)-Bernstein Stancu operators given by (1.2) reduces into (p,q)-Bernstein operators. Also for $p = 1$, (p,q)-Bernstein Stancu operators given by (1.1) turn out to be q-Bernstein Stancu operators.

Let $0 < q < p < 1$ and $x \in [0,\infty)$. We introduce the (p,q)-Stancu-beta operators as follows:

$$S_{uvw,pq}(f,x) =$$

$$\frac{1}{B_{pq}([uvw]_{pq}x,[uvw]_{pq}+3)}\int_0^\infty\int_0^\infty\int_0^\infty \frac{(rst)^{[uvw]_{pq}(x-1)}}{(1+(rst))^{[uvw]_{pq}x+[uvw]_{pq}+3}} f\left(p^{[uvw]_{pq}x}, q^{[uvw]_{pq}x} rst\right) d_{pq}r\, d_{pq}s\, d_{pq}t.$$

Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of beta Stancu operators of $\left(S_{rst,p,q}(f,x)\right)$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st - lim \quad x = 0$, provided that the set

$$K_\epsilon := \left\{(m,n,k) \in \mathbb{N}^3 : \left|S_{rst,p,q}(f,x) - (f,x)\right| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence of beta

Stancu operators. i.e., $\delta(K_\epsilon)=0$. That is,

$$lim_{r,s,t\to\infty}\frac{1}{pqj}\left|\left\{m\le p,n\le q,k\le j:\left|S_{rst,p,q}(f,x)-(f,x)\right|\ge\epsilon\right\}\right|=0.$$

In this case, we write $\delta-limS_{rst,p,q}(f,x)=(f,x)$ or $S_{rst,p,q}(f,x)\to^{S_s}(f,x)$.

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

A triple sequence (real or complex) can be defined as a function $x:\mathbb{N}^3\to\mathbb{R}(\mathbb{C})$, where $\mathbb{N},\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. (Sahiner et al., 2007) and Sahiner and Tripathy (Sahiner and Tripathy, 2008), Esi (Esi, 2014), Esi and Catalbas (Esi and Catalbas, 2014), Esi and Savas (Esi and Savas, 2015), Esi et al. (Esi et al., 2016, 2017a, 2017b), Dutta et al. (Dutta et al., 2013), Subramanian and Esi (Subramanian and Esi, 2015, 2017), Debnath et al. (Debnath et al., 2015) and many others.

A triple sequence $x=(x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k}|x_{mnk}|^{\frac{1}{m+n+k}}<\infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3.

The Borel summability of gradual real numbers is denoted by $(\zeta,X)(\lambda)\in(\mathbb{R})$, and d denotes the supremum metric on $(\zeta,X)(\lambda)\in(\mathbb{R}^3)$. Now let r be nonnegative real number. A Borel summability of rough triple sequence space of greadual real numbers of beta stancu operators of $A_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda)$ of gradual real numbers is $r-$ convergent to a gradul real number $(\zeta,X)(\lambda)$ and we write

$$A_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda)\to^r(\zeta,X)(\lambda)\text{ as }m,n,k\to\infty,$$

provided that for every $\epsilon>0$ there is an integer $m_\epsilon,n_\epsilon,k_\epsilon$ so that

$$d\left(A_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda),(\zeta,X)(\lambda)\right)<r+\epsilon\text{ whenever }m\ge m_\epsilon,n\ge n_\epsilon,k\ge k_\epsilon.$$

The set

$$LIM^rA_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda):$$
$$=\left\{(\zeta,X)(\lambda)\in(\zeta,X)(\lambda)\in(\mathbb{R}^3):A_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda)\to^r(\zeta,X)(\lambda),\quad as\quad m,n,k\to\infty\right\}$$

is called the r-limit set of the Borel summability of rough triple sequence space of gradual real numbers of beta Stancu operators of $\left(A_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda)\right)$.

A Borel summability of rough triple sequence space of beta Stancu operators of gradual real numbers which is divergent can be convergent with a certain roughness degree. For instance, let us define

$$A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda) = \begin{cases} \eta(X), & if\ m,n,k\ are\ odd\ integers, \\ \mu(X), & otherwise \end{cases},$$

where

$$\eta(X) = \begin{cases} X, & if\ \ X \in [0,1], \\ -X+2, & if\ X \in [1,2], \\ 0, & otherwise \end{cases}$$

and

$$\mu(X) = \begin{cases} X-3, & if\ \ X \in [3,4], \\ -X+5, & if\ X \in [4,5], \\ 0, & otherwise \end{cases}.$$

Then we have where

$$LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda) = \begin{cases} \phi, & if\ r < \frac{3}{2}, \\ [\mu - r_1, \eta + r_1], & otherwise \end{cases},$$

where r_1 is nonnegative real number with

$$[\mu - r_1, \eta + r_1] := \left\{ A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda) \in (\zeta, X)(\lambda) \in (\mathbb{R}^3) : \mu - r_1 \leq A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda) \leq \eta + r_1 \right\}.$$

The ideal of rough convergence of a Borel summability of triple sequence space of gradual real numbers of beta Stancu operators can be interpreted as follows:

Let $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda)\right)$ be a convergent triple sequence space of beta stancu operators of gradual real numbers. Assume that $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda)\right)$ cannot be determined exactly for every $(m,n,k) \in \mathbb{N}^3$. That is, $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda)\right)$ cannot be calculated so we can use approximate value of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda)\right)$ for simplicity of calculation. We only know that $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda)\right) \in [\mu_{mnk}, \gamma_{mnk}]$, where $d(\mu_{mnk}, \gamma_{mnk}) \leq r$ for every $(m,n,k) \in \mathbb{N}^3$. The Borel summability of rough triple sequence space of gradul real numbers of beta stancu operators of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)\right)$ satisfying $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)\right) \in [\mu_{mnk}, \gamma_{mnk}]$, for all m,n,k. Then the Borel summability of rough triple sequence space of gradual real numbers of beta stancu operators of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)\right)$ may not be convergent, but the inequality

$$d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), (\zeta, X)(\lambda)\right)$$
$$\leq d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda)\right) + d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda), (\zeta, Y)(\lambda)\right)$$
$$\leq r + d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda), (\zeta, Y)(\lambda)\right)$$

implies that the Borel summability of rough triple sequence space of gradual real numbers of beta Stancu operators of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)\right)$ is r-convergent.

In this paper, we first define the concept of rough convergence of a Borel summability of triple sequence space of beta Stancu operators of gradual real numbers. Also obtain the relation between the set of rough limit and the extreme limit points of a Borel summability of triple sequence space of beta stancu operators of gradul real numbers. We show that the rough limit set of a Borel summability of triple sequence space of gradual real numbers of beta Stancu operators is closed, bounded and convex.

II. Main Results

Theorem 2.1. Let f be a continuous function defined on $\lambda \in (0,1]$. A Borel summability of rough triple sequence of Bernstein-Stancu operators of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)\right)$ of gradual real numbers. If $(\zeta, X)(\lambda) \in LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)$, then

$$diam\left(lim \quad sup \quad A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), (\zeta, X)(\lambda)\right) \leq r$$

and

$$diam\left(lim \quad inf \quad A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), (\zeta, X)(\lambda)\right) \leq r.$$

Proof. We assume that $diam\left(lim \quad sup \quad a_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), (\zeta, X)(\lambda)\right) > r.$

Define $\tilde{\epsilon} := \frac{\left(lim \quad sup \quad A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), (\zeta, X)(\lambda)\right) - r}{2}$. By definition of limit supremem, we have that given $m'_{\tilde{\epsilon}}, n'_{\tilde{\epsilon}}, k'_{\tilde{\epsilon}} \in \mathbb{N}^3$ there exists an $(m, n, k) \in \mathbb{N}^3$ with $m \geq m'_{\tilde{\epsilon}}, n \geq n'_{\tilde{\epsilon}}, k \geq k'_{\tilde{\epsilon}}$ such that $diam\left(lim \quad sup \quad A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), (\zeta, X)(\lambda)\right) \leq \tilde{\epsilon}$. Also, since $A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda) \to^r (\zeta, X)(\lambda)$ as $m, n, k \to \infty$, there is an integer $m''_{\tilde{\epsilon}}, n''_{\tilde{\epsilon}}, k''_{\tilde{\epsilon}}$ so that

$$d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), (\zeta, X)(\lambda)\right) < r + \tilde{\epsilon}$$

whenever $m \geq m''_{\tilde{\epsilon}}, n \geq n''_{\tilde{\epsilon}}, k \geq k''_{\tilde{\epsilon}}$. Let

$$(m_{\tilde{\epsilon}}, n_{\tilde{\epsilon}}, k_{\tilde{\epsilon}}) := max\{(m'_{\tilde{\epsilon}}, n'_{\tilde{\epsilon}}, k'_{\tilde{\epsilon}}), (m''_{\tilde{\epsilon}}, n''_{\tilde{\epsilon}}, k''_{\tilde{\epsilon}})\}.$$

There exists $(m,n,k) \in \mathbb{N}^3$ such that $m \geq m_{\tilde{\epsilon}}, n \geq n_{\tilde{\epsilon}}, k \geq k_{\tilde{\epsilon}}$ and

$$diam\left(lim \quad sup \quad A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda), (\zeta,X)(\lambda)\right)$$

$$\leq (\zeta,X)(\lambda) diam\left(lim \quad sup \quad A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda), A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda)\right)$$

$$+ diam\left(A_{\|.\|_G, S_{rst,p,q}}(\lambda), (\zeta,X)(\lambda)\right)$$

$$< \tilde{\epsilon} + r + \tilde{\epsilon}$$

$$< r + 2\tilde{\epsilon}$$

$$= r + diam\left(lim \quad sup \quad A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda), (\zeta,X)(\lambda)\right) - r$$

$$= diam\left(lim \quad sup \quad A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda), (\zeta,X)(\lambda)\right).$$

The contradiction proves the theorem. Similarly,

$diam\left(lim \quad inf \quad A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda), (\zeta,X)(\lambda)\right) \leq r$ can be proved using definition of limit infimum.

Theorem 2.2. Let f be a continuous function defined on $\lambda \in (0,1]$. A Borel summability of rough triple sequence of Bernstein-Stancu operators of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda)\right)$ of gradual real numbers. If $LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda) \neq \phi$, then we have

$$LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda) \subseteq \left[\left(limsup A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda)\right) - r_1, \left(liminf A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda)\right) + r_1\right].$$

Proof. To prove that $(\zeta,X)(\lambda) \in \left[\left(limsup A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda)\right) - r_1, \left(liminf A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda)\right) + r_1\right]$ for an arbitrary $(\zeta,X)(\lambda) \in LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda)$, i.e.,

$$\left(limsup A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda)\right) - r_1 \leq (\zeta,X)(\lambda) \leq \left(liminf A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda)\right) + r_1.$$

Let us assume that $\left(limsup A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda)\right) - r_1 \leq (\zeta,X)(\lambda)$ does not hold.Thus, there exists an $\alpha_0 \in (0,1]$ such that

$\left(\underline{limsup A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda)^{\alpha_0}}\right) - r_1 > \underline{(\zeta,X)(\lambda)^{\alpha_0}}$ or $\left(\overline{limsup A_{\|.\|_G, S_{mnk}}(\zeta,X)(\lambda)}^{\alpha_0}\right) - r_1 >\leq \overline{(\zeta,X)(\lambda)}^{\alpha_0}$

holds i.e.,

$\left(\underline{limsup A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda)^{\alpha_0}}\right) - \underline{(\zeta,X)(\lambda)^{\alpha_0}} > r_1$ or $\left(\overline{limsup A_{\|.\|_G, S_{rst,p,q}}(\zeta,X)(\lambda)}^{\alpha_0}\right) - \leq \overline{(\zeta,X)(\lambda)}^{\alpha_0} > r_1.$

On the other hand, by theorem 2.1 we have

$$\left|\left(\underline{limsupA_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda)^{\alpha_0}}\right) - \underline{(\zeta,X)(\lambda)^{\alpha_0}}\right| \leq r_1$$

and

$$\left|\left(\overline{limsupA_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda)}^{\alpha_0}\right) - \leq \overline{(\zeta,X)(\lambda)}^{\alpha_0}\right| \leq r_1.$$

We obtain a contradiction. Hence we get $\left(limsupA_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda)\right) - r_1 \leq (\zeta,X)(\lambda)$. By using the similar arguments and get it for second part.

Note 2.3. The converse inclusion in this theorem holds for f be a continuous function defined on $\lambda \in (0,1]$. A Borel summability of rough triple sequence of Bernstein-beta Stancu operators of $\left(A_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda)\right)$ of gradual real numbers, but it may not hold for Borel summability of rough triple sequences of Bernstein-beta Stancu operators of gradual real numbers as in the following example:

Example: Define

$$A_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda) = \begin{Bmatrix} \frac{-1}{2(mnk)}X + 1, & if \quad X \in (0,1], \\ 0, & otherwise \end{Bmatrix}$$

and

$$(\zeta,X)(\lambda) = \begin{Bmatrix} 1, & if \quad X \in (0,1], \\ 0, & otherwise \end{Bmatrix}.$$

Then we have $\left|\overline{(\zeta,X)(\lambda)}^{1} - \overline{A_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda)}^{1}\right| = |1-0| = 1$, i.e.,

$d\left(A_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda),(\zeta,X)(\lambda)\right) \geq 1$ for all $(m,n,k) \in \mathbb{N}^3$. Although the Borel summability of rough triple sequence spaces of Bernstein-beta Stancu operators of gradual real numbers of $\left(A_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda)\right)$ is not convergent to $(\zeta,X)(\lambda)$,

$limsupA_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda)$ and $liminfA_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda)$ of this Borel summability of rough triple sequence space of Bernstein-beta Stancu opeators of gradul real numbers are equal to $(\zeta,X)(\lambda)$. Hence we get $L \in \left[limsupA_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda) - \left(\frac{1}{2}\right)_1, liminfA_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda) + \left(\frac{1}{2}\right)_1\right]$, but $(\zeta,X)(\lambda) \notin LIM^{\frac{1}{2}}A_{\|.\|_G,S_{rst,p,q}}(\zeta,X)(\lambda)$.

Theorem 2.4. Let f be a continuous function defined on $\lambda \in (0,1]$. A Borel summability of rough triple sequence of Bernstein-beta Stancu operators of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)\right)$ of real numbers converges to the gradual real numbers of (f, X), then

$$LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda) = \bar{S}_r\big((\zeta, X)(\lambda)\big) := \{\mu \in (\zeta, X)(\lambda) \in (\mathbb{R}^3): d\big(\mu, (\zeta, X)(\lambda)\big) \leq r\}.$$

Proof. Let $\epsilon > 0$. Since the Borel summability of rough triple sequence space of Bernstein-beta Stancu operators of gradual real numbers of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)\right)$ is convergent to $(\zeta, X)(\lambda)$, there is an integer $m_\epsilon, n_\epsilon, k_\epsilon$ so that

$$d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), (\zeta, X)(\lambda)\right) < \epsilon \text{ whenever } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon.$$

Let $Y \in \bar{S}_r\big((\zeta, X)(\lambda)\big)$, we have

$d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), Y\right) \leq d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), (\zeta, X)(\lambda)\right) + d\big((\zeta, X)(\lambda), Y\big) < \epsilon + r$ for every $m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon$.

Hence we have $Y \in LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)$.

Now let $Y \in LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)$. Hence there is an integer $m_\epsilon^{'}, n_\epsilon^{'}, k_\epsilon^{'}$ so that

$$d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), Y\right) < r + \epsilon$$

whenever $m \geq m_\epsilon^{'}, n \geq n_\epsilon^{'}, k \geq k_\epsilon^{'}$. Let

$$\left(m_\epsilon^{''}, n_\epsilon^{''}, k_\epsilon^{''}\right) := max\{(m_\epsilon, n_\epsilon, k_\epsilon), (m_\epsilon^{'}, n_\epsilon^{'}, k_\epsilon^{'})\}$$

for all $m \geq m_\epsilon^{''}, n \geq n_\epsilon^{''}, k \geq k_\epsilon^{''}$, we obtain

$$d\big(Y, \zeta(X)(\lambda)\big) \leq d\left(Y, A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)\right) + d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), (\zeta, X)(\lambda)\right) < r + \epsilon + \epsilon < r + 2\epsilon.$$

Since ϵ is arbitrary, we have $d\big(Y, (\zeta, X)(\lambda)\big) \leq r$. Hence we get $Y \in \bar{S}_r\big((\zeta, X)(\lambda)\big)$. Thus, if the Borel summability of rough triple sequence space of Bernstein-beta Stancu operators of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)\right)(\lambda) \to^r (\zeta, X)(\lambda)$, then $LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda) = \bar{S}_r\big((\zeta, X)(\lambda)\big)$.

Theorem 2.5. Let f be a continuous function defined on $\lambda \in (0,1]$. A Borel summability of rough triple sequence of gradual real numbers of Bernstein-beta Stancu operators of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)\right)(\lambda)$ and $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda)\right) \in (\zeta, X)(\lambda)(\mathbb{R}^3)$. If $A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda) \to^r (\zeta, X)(\lambda)$ then

$A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda) \to^r (\zeta, Y)(\lambda)$ and $d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda)\right) \leq r$ for every $(m, n, k) \in \mathbb{N}^3$.

Proof. Assume that $A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda) \to^r (\zeta, Y)(\lambda)$, as $m, n, k \to \infty$ and $d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda)\right) \leq r$ for every $(m, n, k) \in \mathbb{N}^3$. We have $A_{\|.\|_G, S_{mnk}}(\zeta, Y)(\lambda) \to^r (\zeta, Y)(\lambda)$, as $m, n, k \to \infty$ means that for every $\epsilon > 0$ there exists an $m_\epsilon, n_\epsilon, k_\epsilon$ such that $d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda), (\zeta, Y)(\lambda)\right) < \epsilon$ for all $m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon$.

If the in equality $d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda)\right) \leq r$ yields then

$$d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), (\zeta, X)(\lambda)\right) \leq$$

$$d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda)\right) + d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda), (\zeta, Y)(\lambda)\right) < r + \epsilon \text{ for all } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon.$$

Hence the Borel summability of rough triple sequence space of Bernstein-beta Stancu operators of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)\right)$ is r-convergent to the gradual real numbers $(\zeta, X)(\lambda)$.

Theorem 2.6. Let f be a continuous function defined on $\lambda \in (0,1]$. A Borel summability of rough triple sequence of Bernstein-beta Stancu operators of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)\right)$ of gradual real numbers and the diameter of an r-limit set is not greater than $3r$.

Proof. We have to prove that

$$sup\left\{d(W, Z): W, Y, Z \in LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)\right\} \leq 3r.$$

Assume on the contrary that

$$sup\left\{d(W, Z): W, Y, Z \in LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)\right\} > 3r.$$

By this assumption, there exists, $W, Y, Z \in LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)$ satisfying $\lambda := d(W, Z) > 3r$. For an arbitrary $\epsilon \in \left(0, \frac{\lambda}{3} - r\right)$, we have

$$\exists\left(m_\epsilon^{'}, n_\epsilon^{'}, k_\epsilon^{'}\right) \in \mathbb{N}^3: \forall m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon \Longrightarrow d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), W\right) \leq r + \epsilon,$$

$$\exists\left(m_\epsilon^{''}, n_\epsilon^{''}, k_\epsilon^{''}\right) \in \mathbb{N}^3: \forall m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon \Longrightarrow d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), Y\right) \leq r + \epsilon,$$

$$\exists\left(m_\epsilon^{'''}, n_\epsilon^{'''}, k_\epsilon^{'''}\right) \in \mathbb{N}^3: m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon \Longrightarrow d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), Z\right) \leq r + \epsilon.$$

Define $(m_\epsilon, n_\epsilon, k_\epsilon) := max\{(m_\epsilon^{'}, n_\epsilon^{'}, k_\epsilon^{'}), (m_\epsilon^{''}, n_\epsilon^{''}, k_\epsilon^{''}), (m_\epsilon^{'''}, n_\epsilon^{'''}, k_\epsilon^{'''})\}$.

Thus we get

$$d(W,Z) \leq d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), W\right) + d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), Y\right) + d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), Z\right)$$
$$< (r+\epsilon) + (r+\epsilon) + (r+\epsilon)$$
$$< 3(r+\epsilon)$$
$$< 3r + 3\left(\frac{\lambda}{3} - r\right) < 3r + \lambda - 3r$$
$$= \lambda \text{ for all } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon$$

which contradicts to the fact that $\lambda = d(W,Z)$.

Theorem 2.7. Let f be a continuous function defined on $\lambda \in (0,1]$. A Borel summability of rough triple sequence of Bernstein-beta Stancu operators of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)\right)$ of gradual real numbers is analytic if and only if there exists an $r \geq 0$ such that $LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda) \neq \phi$.

Proof. Necessity: Let the Borel summability of rough triple sequence space of Bernstein-beta Stancu operators of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)\right)$ be a analytic sequence and

$$s := sup\left\{d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)^{1/m+n+k}(\lambda), 0\right) : (m,n,k) \in \mathbb{N}^3\right\} < \infty.$$

Then we have $0 \in LIM^s A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)$, i.e., $LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda) \neq \phi$, where $r = s$.

Sufficiency: If $LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda) \neq \phi$ for some $r \geq 0$, then there exists $(\zeta, X)(\lambda) \in LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)$. By definition, for every $\epsilon > 0$ there is an integer $(m_\epsilon, n_\epsilon, k_\epsilon)$ so that

$$d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), (\zeta, X)(\lambda)\right) < r + \epsilon \text{ whenever } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon.$$

Define

$$t = t(\epsilon) := Max\{d((\zeta, X)(\lambda), 0), d(S_{111,p,q}(\zeta, X)(\lambda), 0), \cdots, d(S_{r_\epsilon s_\epsilon t_\epsilon, p,q}(\zeta, X)(\lambda), 0), r + \epsilon\}.$$

Then we have

$$A_{\|.\|_G, S_{rst,p,q}} \in \{\mu \in (\zeta, X)(\lambda)(\mathbb{R}^3) : d(\mu, 0) \leq t + r + \epsilon\} \text{ for every } (m,n,k) \in \mathbb{N}^3,$$

which proves the boundedness of the Borel summability of rough triple sequence space of Bernstein-beta Stancu operators of gradual real numbers of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)\right)$.

Theorem 2.8. Let f be a continuous function defined on $\lambda \in (0,1]$. A Borel summability of rough triple sequence of Bernstein-beta Stancu operators of $\left(A_{\|.\|_G, S_{u_m v_n w_k, p,q}}(\zeta, X)(\lambda)\right)$ of real numbers is a sub sequence of a Borel summability of rough triple sequence space of Bernstein-beta Stancu operators of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)\right)$, then $LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda) \subset LIM^r A_{\|.\|_G, S_{u_m v_n w_k, p,q}}(\zeta, X)(\lambda)$.

Proof. Omitted.

Theorem 2.9. Let f be a continuous function defined on $\lambda \in (0,1]$. A Borel summability of rough triple sequence of Bernstein-beta Stancu operators of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)\right)$ of gradual real numbers, for all $r \geq 0$, the r-limit set $LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)$ of an arbitrary Borel summability of rough triple sequence space of Bernstein-beta Stancu operators of gradual real numbers of $A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)$ is closed.

Proof. Let $(Y_{mnk}) \subset LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda)$ and $A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda) \to (\zeta, Y)(\lambda)$ as $m, n, k \to \infty$. Let $\epsilon > 0$. Since the Borel summability of rough triple sequence space of Bernstein-beta Stancu operators of gradual real numbers of $\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda)\right) \to^r (\zeta, Y)(\lambda)$ there is an integer $i_\epsilon j_\epsilon \ell_\epsilon$ so that

$d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, Y)(\lambda), (\zeta, Y)(\lambda)\right) < \frac{\epsilon}{2}$ whenever $m \geq i_\epsilon, n \geq j_\epsilon, k \geq \ell_\epsilon$.

Since $S_{i_\epsilon j_\epsilon \ell_\epsilon, p,q}(\zeta, Y)(\lambda) \in LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)$, there is an integer $m_\epsilon n_\epsilon k_\epsilon$ so that

$$d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), A_{\|.\|_G, S_{i_\epsilon j_\epsilon \ell_\epsilon, p,q}}(\zeta, Y)(\lambda)\right) < r + \frac{\epsilon}{2} \text{ whenever } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon.$$

Therefore, we have

$$d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), (\zeta, X)(\lambda)\right) \leq d\left(A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda), A_{\|.\|_G, S_{i_\epsilon j_\epsilon \ell_\epsilon, p,q}}(\zeta, Y)(\lambda)\right) < r + \frac{\epsilon}{2} + \frac{\epsilon}{2} = r + \epsilon$$

for every $m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon$.

Hence $L \in LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)$ implies that the set $LIM^r A_{\|.\|_G, S_{rst,p,q}}(\zeta, X)(\lambda)$ is closed.

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