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Research Article

More Correct Berezin Symbol Inequalities

Hamdullah Başaran[a,*], Mehmet Gürdal[a]
[a]Department of Mathematics, Suleyman Demirel University, Isparta, Turkey.

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*Corresponding Author: Hamdullah Başaran
(07hamdullahbasaran@gmail.com)

Abstract: The purpose of this research is to show bounds for some Berezin number inequalities in an innovative approach. Some inequalities have been proven using the improvement of the Hermite-Hadamard inequality. These inequalities are a refined version of Huban et al.'s inequalities (Huban et al., 2021b; Huban et al., 2022a) and Başaran et al.'s inequalities (Başaran et al., 2022). Finally, we prove last three theorems by applying the method of Cartesian decomposition.

Keywords: Berezin number, Berezin norm, Hermite-Hadamard inequality, Operator convex function, Jensen inequality.

I. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with an inner product $\langle .,. \rangle$ and corresponding norm $\|.\|$. Let $\mathcal{L}(\mathcal{H})$ define the C^* -algebra of all bounded linear operators on $\mathcal{H}$. The numerical range and numerical radius of $T \in \mathcal{L}(\mathcal{H})$ are denoted by

$$W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H} \text{ } and \text{ } \|x\| = 1\} \text{ } and \text{ } w(T) = sup\{|\langle Tx, x \rangle| : x \in \mathcal{H} \text{ } and \text{ } \|x\| = 1\},$$

respectively. The absolute value of positive operator is denoted by $|T| = (T^*T)^{\frac{1}{2}}$.

Recall that a reproducing kernel Hilbert space (briefly, RKHS) $\mathcal{H} = \mathcal{H}(\Omega)$ is a complex Hilbert space on a (nonempty) Ω, which has the property that point evalutians are continuous for each $\xi \in \Omega$ there is an unique element $k_\xi \in \mathcal{H}$ such that $f(\xi) = \langle f, k_\xi \rangle$, for all $f \in \mathcal{H}$. The family $\{k_\xi : \xi \in \Omega\}$ is called the reproducing kernel of $\mathcal{H}$. If $\{e_n\}_{n\geq 0}$ is an orthonormal basis for reproducing kernel Hilbert space, the reproducing kernel is shown by $k_\xi = \sum_{n=0}^{\infty} \overline{e_n(\xi)}\, e_n(z)$; (see, (Aronzajn, 1950) and (Berezin, 1972)). For $\xi \in \Omega$, $\hat{k}_\xi = \frac{k_\xi}{\|k_\xi\|}$ is called the normalized reproducing kernel.

For $T \in \mathcal{L}(\mathcal{H})$, the function $\tilde{T}$ defined on Ω by $\tilde{T}(\xi) = \langle T\hat{k}_\xi, \hat{k}_\xi \rangle$ is the Berezin symbol (or Berezin transform) of T. Berezin symbol firstly has been introduced by Berezin (Berezin, 1972). The Berezin set and Berezin number of the operator T are defined by

$$Ber(T) = \{\tilde{T}(\xi) : \xi \in \Omega\} \; and \; ber(T) = \left\{sup\left|\tilde{T}(\xi)\right| : \xi \in \Omega\right\},$$

respectively (see, (Karaev,2006; Karaev, 2013). In some recent works, several Berezin radius inequalities have been examined by authors (Başaran and Gürdal, 2021), (Başaran and Gürdal, 2023a), (Başaran and Gürdal, 2023b), (Chalender et al., 2012), (Garayev et al., 2020), (Garayev and Alomari, 2021), (Garayev et al., 2021), (Gürdal and Başaran, 2022).

We also define the following so-called Berezin norm of operators $T \in \mathcal{L}(\mathcal{H})$:

$$\|T\|_{Ber} = sup_{\xi\in\Omega}\left\|T\hat{k}_\xi\right\|.$$

It is easy to see that actually $\|T\|_{Ber}$ determines a new operator norm in $\mathcal{L}\big(\mathcal{H}(\Omega)\big)$: (since the set of reproducing kernels $\left\{\hat{k}_\xi : \xi \in \Omega\right\}$ span the space $\mathcal{H}(\Omega)$). It is also trivial that $ber(T) \leq \|T\|_{Ber} \leq \|T\|$.

In a RKHS, the Berezin range of an operator T is a subset of numerical range of T, i.e.,

$$Ber(T) \subseteq W(T).$$

Hence,

$$ber(T) \leq w(T).$$

There are interesting properties of numerical range. For example, it is well known that the spectrum of an operator is contained in the closure of its numerical range. For basic properties numerical radius, we refer to (Alomari, 2021), (Aici et al., 2023), (Bhunia and Paul, 2021), (Bhunia and Paul, 2023), (Dragomir, 2009), (Gustanson and Rao, 1997), (Jana et al., 2023), (Kian, 2014), (Kian and Alomari, 2022), (Qawasmeh et al., 2023).

It is well known that

$$\frac{\|T\|}{2} \leq w(T) \leq \|T\| \tag{1.1}$$

and

$$ber(T) \leq w(T) \leq \|T\|, \tag{1.2}$$

for any $T \in \mathcal{L}(\mathcal{H})$. Also, Berezin range and Berezin radius of operators are numerical characteristics of operators on the RKHS, which are shown Karaev in (Karaev, 2006). For the basic properties and facts on these new concepts see (Karaev, 2013).

In 2021, Huban et al. (Huban et al., 2021a) obtained the following inequalities

$$\frac{1}{4}\|T^*T + TT^*\|_{ber} \leq ber(T) \leq \frac{1}{2}\|T^*T + TT^*\|_{ber} \tag{1.3}$$

and

$$ber^r(T^*R) \leq \frac{1}{2}\||T|^{2r} + |R|^{2r}\|_{ber} \tag{1.4}$$

for $r \geq 1$ and for any $T, R \in \mathcal{L}(\mathcal{H})$. One year later, the same authors (Huban et al., 2021b; Huban et al., 2022a) found the inequalities

$$ber(T) \leq \frac{1}{2}\||T| + |T^*|\|_{ber} \leq \frac{1}{2}\left(\|T\|_{ber} + \|T^2\|_{ber}^{1/2}\right) \tag{1.5}$$

and

$$f\left(ber(T)\right) \leq \frac{1}{2}\left\|f\left(\frac{3|T|+|T^*|}{4}\right) + f\left(\frac{|T|+3|T^*|}{4}\right)\right\|_{ber} \tag{1.6}$$

for any $T \in \mathcal{L}(\mathcal{H})$ and all increasing convex function $f: [0, \infty) \to [0, \infty)$.

Başaran et al. (Başaran et al., 2022) demonstrated some new Berezin number inequalities. One these inequalities, the well-known Hermite-Hadamard inequality was utilized to perform the following result.

$$f\left(ber(T)\right) \leq \left\|\int_0^1 f\left((1-t)|T| + t|T^*|\right)\right\|_{ber} \leq \frac{1}{2}\|f(T) + f(T^*)\|_{ber} \tag{1.7}$$

for any $T \in \mathcal{L}(\mathcal{H})$ and all increasing operator convex function $f: [0, \infty) \to [0, \infty)$.

On the other hand, Gürdal and Başaran (Gürdal and Başaran, 2023c) proved that

$$ber^2(T) \leq \frac{1}{12}\||T| + |T^*|\|_{ber}^2 + \frac{1}{3}ber(T)\||T| + |T^*|\|_{ber} \tag{1.8}$$

$$\leq \frac{1}{6}\||T|^2 + |T^*|^2\|_{ber} + \frac{1}{3}ber(T)\||T| + |T^*|\|_{ber}$$

$$\leq \frac{1}{4}\||T| + |T^*|\|_{ber}^2$$

for any $T \in \mathcal{L}(\mathcal{H})$. In the same work, a improvement of (1.4) was shown, as follows:

$$ber^{2r}(T^*R) \leq \frac{1}{4}\mu\||T|^{2r} + |R|^{2r}\|_{ber}^2 + \frac{1}{2}(1-\mu)ber^r(T)\||T|^{2r} + |R|^{2r}\|_{ber} \tag{1.9}$$

$$\leq \frac{1}{2}\mu\||T|^{4r} + |R|^{4r}\|_{ber} + \frac{1}{2}(1-\mu)ber^r(T)\||T|^{2r} + |R|^{2r}\|_{ber}$$

$$\leq \frac{1}{2}\||T|^{4r} + |R|^{4r}\|_{ber}$$

for any $T \in \mathcal{L}(\mathcal{H})$, $r \geq 1$ and $\mu \in [0,1]$. If $r = 1$, then we can see that

$$ber^2(T^*R) \leq \frac{1}{4}\mu\||T|^2 + |R|^2\|_{ber}^2 + \frac{1}{2}(1-\mu)ber(T)\||T|^2 + |R|^2\|_{ber} \tag{1.10}$$

$$\leq \frac{1}{2}\mu\||T|^4 + |R|^4\|_{ber} + \frac{1}{2}(1-\mu)ber(T)\||T|^2 + |R|^2\|_{ber}$$

$$\leq \frac{1}{2}\||T|^4 + |R|^4\|_{ber}.$$

The purpose of this study is to demonstrate bounds for several Berezin number inequalities. Some inequalities have been demonstrated using the Hermite-Hadamard inequality. These inequalities are a revised version of Huban et al. (Huban et al., 2021b; Huban et al., 2022a) and Başaran et al. (Başaran et al., 2022). Finally, we use the Cartesian decomposition approach to show the following three theorems.

We now present the lemma that is needed for the continuation.

Lemma 1.1 (Furuta et al., 1995, Theorem 1.4): *If* $T \in \mathcal{L}^+(\mathcal{H})$*, then*

$$\langle Tx, x\rangle^r \leq \langle T^r x, x\rangle, r \geq 1, \tag{1.11}$$

for any vector $x \in \mathcal{H}$. *If* $0 \leq r \leq 1$*, then the inequality* (1.11) *is reversed.*

Lemma 1.2 (Kittaneh, 1988): *Let* $T \in \mathcal{L}(\mathcal{H})$*. Then*

$$\langle Tx, y\rangle^2 \leq \langle |T|^{2r}x, x\rangle\langle T^{2(1-r)}y, y\rangle, 0 \leq r \leq 1, \tag{1.12}$$

for any vectors x, y $\in \mathcal{H}$.

Lemma 1.3 (Furuta et al., 1995, Theorem 1.2): *Let* T *be a selfadjoint operator whose spectrum* $T \in [m, M]$ *for scalar* $m \leq M$. *If* $f(t)$ *is a convex function on* $[m, M]$, *then*

$$f(\langle Tx, x\rangle) \leq \langle f(T)x, x\rangle \tag{1.13}$$

for any unit vector x $\in \mathcal{H}$.

II. Main Results

Now, let's prove the first theorem.

Theorem 2.1: *Let* $\mathcal{H} = \mathcal{H}(\Omega)$ *be an RKHS. If* $f: [0, \infty) \to [0, \infty)$ *is an increasing convex function, then*

$$f(ber(T)) \leq \frac{1}{2}\left\| f\left(\frac{2|T|+|T^*|}{3}\right) + f\left(\frac{|T|+2|T^*|}{3}\right)\right\|_{ber} \tag{2.1}$$

for any operator $T \in \mathcal{L}(\mathcal{H})$.

Proof: Let f is an increasing operator convex function and let $\hat{k}_\xi$ be a normalized reproducing kernel. Then we get

$$f\left(\left|\langle T\hat{k}_\xi, \hat{k}_\xi\rangle\right|\right) \leq f\left(\sqrt{\langle |T|\hat{k}_\xi, \hat{k}_\xi\rangle\langle |T^*|\hat{k}_\xi, \hat{k}_\xi\rangle}\right)$$

$$\leq f\left(\langle\left(\frac{|T|+|T^*|}{2}\right)\hat{k}_\xi, \hat{k}_\xi\rangle\right) \text{ (by A.M-G.M inequality)}$$

$$= f\left(\frac{1}{2}\left[\langle\left(\frac{2|T|+|T^*|}{3}\right)\hat{k}_\xi, \hat{k}_\xi\rangle + \langle\left(\frac{|T|+2|T^*|}{3}\right)\hat{k}_\xi, \hat{k}_\xi\rangle\right]\right)$$

$$\leq \frac{1}{2}\left[f\left(\langle\left(\frac{2|T|+|T^*|}{3}\right)\hat{k}_\xi, \hat{k}_\xi\rangle\right) + f\left(\langle\left(\frac{|T|+2|T^*|}{3}\right)\hat{k}_\xi, \hat{k}_\xi\rangle\right)\right]$$

(by the Jensen inequality)

$$\leq \frac{1}{2}\left[\langle f\left(\frac{2|T|+|T^*|}{3}\right)\hat{k}_\xi, \hat{k}_\xi\rangle + \langle f\left(\frac{|T|+2|T^*|}{3}\right)\hat{k}_\xi, \hat{k}_\xi\rangle\right] \text{ (by (1.13))}$$

$$= \frac{1}{2}\langle\left[f\left(\frac{2|T|+|T^*|}{3}\right) + f\left(\frac{|T|+2|T^*|}{3}\right)\right]\hat{k}_\xi, \hat{k}_\xi\rangle$$

Taking the supremum over $\xi \in \Omega$ in the above inequality, so we reach the required result.

Corollary 2.2: *If $f:[0,\infty) \to [0,\infty)$ is an increasing convex function, then*

$$ber^r(T) \leq \frac{1}{2}\left\| \left(\frac{2|T| + |T^*|}{3}\right)^r + \left(\frac{|T| + 2|T^*|}{3}\right)^r \right\|_{ber}$$

for any operator $T \in \mathcal{L}(\mathcal{H})$. In a particular case,

$$ber^2(T) \leq \frac{1}{18}\|(2|T| + |T^*|)^2 + (|T| + 2|T^*|)^2\|_{ber}.$$

A non-trivial improvement (2.1) can be considered in the following result.

Theorem 2.3: *If $f:[0,\infty) \to [0,\infty)$ is an increasing convex function, then*

$$f\big(ber(T)\big) \leq f\left\|\int_0^1 f\left((1-t)\left(\frac{2|T|+|T^*|}{3}\right) + t\left(\frac{|T|+2|T^*|}{3}\right)\right)dt\right\|_{ber} \tag{2.2}$$

$$\leq \frac{1}{2}\left\|f\left(\frac{2|T|+|T^*|}{3}\right) + f\left(\frac{|T|+2|T^*|}{3}\right)\right\|_{ber}$$

$$\leq \left\|\frac{f(|T|)+f(|T^*|)}{2}\right\|_{ber},$$

for any operator $T \in \mathcal{L}(\mathcal{H})$.

Proof: Let $\xi \in \Omega$ be an arbitrary and let f be an increasing operator convex function. Then, we get

$$f\big(|\langle T\hat{k}_\xi, \hat{k}_\xi\rangle|\big) \leq f\left(\sqrt{\langle |T|\hat{k}_\xi, \hat{k}_\xi\rangle \langle |T^*|\hat{k}_\xi, \hat{k}_\xi\rangle}\right)$$

$$\leq f\left(\langle\left(\frac{|T|+|T^*|}{2}\right)\hat{k}_\xi, \hat{k}_\xi\rangle\right) \text{ (by A.M-G.M inequality)}$$

$$= f\left(\frac{1}{2}\left[\langle\left(\frac{2|T|+|T^*|}{3}\right)\hat{k}_\xi, \hat{k}_\xi\rangle + \langle\left(\frac{|T|+2|T^*|}{3}\right)\hat{k}_\xi, \hat{k}_\xi\rangle\right]\right)$$

$$\leq \int_0^1 f\left((1-t)\langle\left(\frac{2|T|+|T^*|}{3}\right)\hat{k}_\xi, \hat{k}_\xi\rangle + t\langle\left(\frac{|T|+2|T^*|}{3}\right)\hat{k}_\xi, \hat{k}_\xi\rangle\right)dt$$

(by Hermite-Hadamard inequality)

$$\leq \int_0^1 f\left(\langle(1-t)\left(\frac{2|T|+|T^*|}{3}\right)\hat{k}_\xi,\hat{k}_\xi\rangle+\langle t\left(\frac{|T|+2|T^*|}{3}\right)\hat{k}_\xi,\hat{k}_\xi\rangle\right)dt. \quad (2.3)$$

On the other hand,

$$\int_0^1 f\left(\langle(1-t)\left(\frac{2|T|+|T^*|}{3}\right)\hat{k}_\xi,\hat{k}_\xi\rangle+\langle t\left(\frac{|T|+2|T^*|}{3}\right)\hat{k}_\xi,\hat{k}_\xi\rangle\right)dt$$

$$\leq \int_0^1 f\left(\langle\left[(1-t)\left(\frac{2|T|+|T^*|}{3}\right)+t\left(\frac{|T|+2|T^*|}{3}\right)\right]\hat{k}_\xi,\hat{k}_\xi\rangle\right)dt$$

$$\leq \int_0^1 \langle f\left((1-t)\left(\frac{2|T|+|T^*|}{3}\right)+t\left(\frac{|T|+2|T^*|}{3}\right)\right)\hat{k}_\xi,\hat{k}_\xi\rangle\, dt \text{ (by (1.13))}$$

$$\leq \int_0^1 \langle\left((1-t)f\left(\frac{2|T|+|T^*|}{3}\right)+tf\left(\frac{|T|+2|T^*|}{3}\right)\right)\hat{k}_\xi,\hat{k}_\xi\rangle\, dt$$

(by Jensen inequality)

$$\leq \langle\left(\int_0^1\left((1-t)f\left(\frac{2|T|+|T^*|}{3}\right)+tf\left(\frac{|T|+2|T^*|}{3}\right)\right)dt\right)\hat{k}_\xi,\hat{k}_\xi\rangle$$

$$\leq \frac{1}{2}\left[\langle\left[f\left(\frac{2|T|+|T^*|}{3}\right)+f\left(\frac{|T|+2|T^*|}{3}\right)\right]\hat{k}_\xi,\hat{k}_\xi\rangle\right]$$

$$\leq \langle\left[\frac{f(|T|)+f(|T^*|)}{2}\right]\hat{k}_\xi,\hat{k}_\xi\rangle. \quad (2.4)$$

By combining (2.3) and (2.4), we have

$$f\left(\left|\langle T\hat{k}_\xi,\hat{k}_\xi\rangle\right|\right)\leq \langle\left(\int_0^1\left((1-t)f\left(\frac{2|T|+|T^*|}{3}\right)+tf\left(\frac{|T|+2|T^*|}{3}\right)\right)dt\right)\hat{k}_\xi,\hat{k}_\xi\rangle$$

$$\leq \frac{1}{2}\left[\langle\left[f\left(\frac{2|T|+|T^*|}{3}\right)+f\left(\frac{|T|+2|T^*|}{3}\right)\right]\hat{k}_\xi,\hat{k}_\xi\rangle\right]$$

$$\leq \langle\left[\frac{f(|T|)+f(|T^*|)}{2}\right]\hat{k}_\xi,\hat{k}_\xi\rangle.$$

Taking the supremum over $\xi \in \Omega$ in the above inequality, we reach

$$f\left(ber(T)\right)\leq f\left\|\int_0^1 f\left((1-t)\left(\frac{2|T|+|T^*|}{3}\right)+t\left(\frac{|T|+2|T^*|}{3}\right)\right)dt\right\|_{ber}$$

$$\leq \frac{1}{2}\left\|f\left(\frac{2|T|+|T^*|}{3}\right)+f\left(\frac{|T|+2|T^*|}{3}\right)\right\|_{ber}$$

$$\leq \left\| \frac{f(|T|)+f(|T^*|)}{2} \right\|_{ber}.$$

Corollary 2.4: *Let* $T \in \mathcal{L}(\mathcal{H})$. *Then,*

$$ber^r(T) \leq \left\| \int_0^1 \left((1-t)\left(\frac{2|T|+|T^*|}{3}\right) + t\left(\frac{|T|+2|T^*|}{3}\right)\right)^r dt \right\|_{ber}$$

$$\leq \frac{1}{2}\left\| \left(\frac{2|T|+|T^*|}{3}\right)^r + \left(\frac{|T|+2|T^*|}{3}\right)^r \right\|_{ber} \tag{2.5}$$

$$\leq \left\| \frac{|T|^r+|T^*|^r}{2} \right\|_{ber}$$

for all $1 \leq r \leq 2$. *In particular case,*

$$ber^2(T) \leq \left\| \int_0^1 \left((1-t)\left(\frac{2|T|+|T^*|}{3}\right) + t\left(\frac{|T|+2|T^*|}{3}\right)\right)^2 dt \right\|_{ber}$$

$$\leq \frac{1}{18}\|(2|T|+|T^*|)^2 + (|T|+2|T^*|)^2\|_{ber}$$

$$\leq \left\| \frac{|T|^2+|T^*|^2}{2} \right\|_{ber}.$$

Proof: The result follows by applying the increasing operator convex function $f(t) = t^r$, $1 \leq r \leq 2$, to inequality (2.3). For particular case, putting $r = 2$ in (2.3).

Theorem 2.5: *If* $f: [0,\infty) \to [0,\infty)$ *is an increasing convex function, then*

$$f\big(ber(V^*R)\big) \leq f\left\| \int_0^1 f\left((1-t)\left(\frac{2|V|^2+|R|^2}{3}\right) + t\left(\frac{|V|^2+2|R|^2}{3}\right)\right) dt \right\|_{ber}$$

$$\leq \frac{1}{2}\left\| f\left(\frac{2|V|^2+|R|^2}{3}\right) + f\left(\frac{|V|^2+2|R|^2}{3}\right) \right\|_{ber}$$

$$\leq \left\| \frac{f(|V|^2)+f(|R|^2)}{2} \right\|_{ber},$$

for any operator $V, R \in \mathcal{L}(\mathcal{H})$.

Proof: Let $\xi \in \Omega$ be an arbitrary. Then, we get

$$f\left(\left|\langle V^{*}R\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle\right|\right) \leq f\left(\left|\langle R\widehat{k}_{\xi},V\widehat{k}_{\xi}\rangle\right|\right)$$

$$\leq f\left(\left\|V\widehat{k}_{\xi}\right\|\left\|R\widehat{k}_{\xi}\right\|\right)$$

$$\leq f\left(\langle |V|^{2}\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle^{1/2}\langle |R|^{2}\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle^{1/2}\right)$$

$$\leq f\left(\frac{\langle |V|^{2}\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle+\langle |R|^{2}\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle}{2}\right) \text{ (by A.M-G.M inequality)}$$

$$\leq f\left(\frac{\langle (|V|^{2}+|R|^{2})\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle}{2}\right)$$

$$= f\left(\frac{1}{2}\left[\langle\left(\frac{2|V|^{2}+|R|^{2}}{3}\right)\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle+\langle\left(\frac{|V|^{2}+2|R|^{2}}{3}\right)\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle\right]\right)$$

$$\leq \int_{0}^{1} f\left((1-t)\langle\left(\frac{2|V|^{2}+|R|^{2}}{3}\right)\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle+t\langle\left(\frac{|V|^{2}+2|R|^{2}}{3}\right)\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle\right)dt$$

(by Hermite-Hadamard inequality)

$$\leq \int_{0}^{1} f\left(\langle(1-t)\left(\frac{2|V|^{2}+|R|^{2}}{3}\right)\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle+\langle t\left(\frac{|V|^{2}+2|R|^{2}}{3}\right)\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle\right)dt. \tag{2.6}$$

On the other hand,

$$\int_{0}^{1} f\left(\langle(1-t)\left(\frac{2|V|^{2}+|R|^{2}}{3}\right)\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle+\langle t\left(\frac{|V|^{2}+2|R|^{2}}{3}\right)\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle\right)dt$$

$$\leq \int_{0}^{1} f\left(\langle\left[(1-t)\left(\frac{2|V|^{2}+|R|^{2}}{3}\right)+t\left(\frac{|V|^{2}+2|R|^{2}}{3}\right)\right]\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle\right)dt$$

$$\leq \int_{0}^{1} \langle f\left((1-t)\left(\frac{2|V|^{2}+|R|^{2}}{3}\right)+t\left(\frac{|V|^{2}+2|R|^{2}}{3}\right)\right)\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle\, dt \text{ (by (1.13))}$$

$$\leq \int_{0}^{1} \langle\left((1-t)f\left(\frac{2|V|^{2}+|R|^{2}}{3}\right)+tf\left(\frac{|V|^{2}+2|R|^{2}}{3}\right)\right)\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle\, dt$$

(by Jensen inequality)

$$\leq \langle\left(\int_{0}^{1}\left((1-t)f\left(\frac{2|V|^{2}+|R|^{2}}{3}\right)+tf\left(\frac{|V|^{2}+2|R|^{2}}{3}\right)\right)dt\right)\widehat{k}_{\xi},\widehat{k}_{\xi}\rangle$$

$$\leq \frac{1}{2}\left[\left\langle\left[f\left(\frac{2|V|^2+|R|^2}{3}\right)+f\left(\frac{|V|^2+2|R|^2}{3}\right)\right]\hat{k}_\xi,\hat{k}_\xi\right\rangle\right]$$

$$\leq \left\langle\left[\frac{f|V|^2+f(|R|^2)}{2}\right]\hat{k}_\xi,\hat{k}_\xi\right\rangle. \tag{2.7}$$

By combining (2.6) and (2.7), we have

$$f\left(\left|\langle V^*R\hat{k}_\xi,\hat{k}_\xi\rangle\right|\right) \leq \left\langle\left(\int_0^1\left((1-t)f\left(\frac{2|V|^2+|R|^2}{3}\right)+tf\left(\frac{|V|^2+2|R|^2}{3}\right)\right)dt\right)\hat{k}_\xi,\hat{k}_\xi\right\rangle$$

$$\leq \frac{1}{2}\left[\left\langle\left[f\left(\frac{2|V|^2+|R|^2}{3}\right)+f\left(\frac{|V|^2+2|R|^2}{3}\right)\right]\hat{k}_\xi,\hat{k}_\xi\right\rangle\right]$$

$$\leq \left\langle\left[\frac{f(|V|^2)+f(|R|^2)}{2}\right]\hat{k}_\xi,\hat{k}_\xi\right\rangle.$$

Taking the supremum over $\xi \in \Omega$ in the above inequality, we reach

$$f\left(ber(V^*R)\right) \leq f\left\|\int_0^1 f\left((1-t)\left(\frac{2|V|^2+|R|^2}{3}\right)+t\left(\frac{|V|^2+2|R|^2}{3}\right)\right)dt\right\|_{ber}$$

$$\leq \frac{1}{2}\left\|f\left(\frac{2|V|^2+|R|^2}{3}\right)+f\left(\frac{|V|^2+2|R|^2}{3}\right)\right\|_{ber}$$

$$\leq \left\|\frac{f(|V|^2)+f(|R|^2)}{2}\right\|_{ber}.$$

We conclude by proposing many important enhancements to Berezin number inequalities. Huban et al. (Huban et al., 2022b) and Başaran et al. (Başaran et al., 2022) introduced new forms of Berezin number inequalities in RKHS, among others. Huban et al. (Huban et al., 2022b) demonstrated their findings using the traditional Hermite-Hadamard inequality and its version. We enhance and extend these inequalities in light of Alomari's improvement and extension of the Hermite-Hadamard inequality (Alomari, 2017).

Theorem 2.6: *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be an RKHS. Let $\phi:\mathcal{L}(\mathcal{H})\to\mathcal{L}(\mathcal{H})$ be a positive unital linear map and $T\in\mathcal{L}(\mathcal{H})$. If $f:[0,\infty)\to[0,\infty)$ is an increasing convex function, then*

$$f\left(ber^2(T)\right) \leq f\left(\frac{1}{2}\|\phi(|T|^2+|T^*|^2)\|_{ber}\right)$$

$$\leq \frac{1}{2}f\left(\left\|\phi\left(\frac{3|T|^2+|T^*|^2}{4}\right)+\phi\left(\frac{|T|^2+3|T^*|^2}{4}\right)\right\|_{ber}\right)$$

$$\leq sup_{\xi\in\Omega}\int_0^1 f\left(\left\|\phi^{\frac{1}{2}}\big((1-t)|T|^2+t|T^*|^2\big)\hat{k}_\xi\right\|^2\right)dt$$

$$\leq f\left(\left\|\phi\left(\frac{|T|^2+|T^*|^2}{2}\right)\right\|_{ber}\right)+\frac{1}{2}\phi(\|f(|T|^2)+f(|T^*|^2)\|_{ber})$$

$$\leq \frac{1}{2}\left\|\phi\big(f(|T|^2)+f(|T^*|^2)\big)\right\|_{ber}$$

for any $\xi\in\Omega$.

Proof: In (Alomari et al., 2022), Alomari shown the following improvement of the classical Hermite-Hadamard inequality:

$$\frac{(b-a)}{2}\left[h\left(\frac{3a+b}{4}\right)+h\left(\frac{a+3b}{4}\right)\right]\leq\int_a^b h(t)dt \tag{2.8}$$

$$\leq\frac{(b-a)}{2}\left[h\left(\frac{a+b}{2}\right)+\frac{h(a)+h(b)}{2}\right]$$

for every convex function h: $[0,\infty)\to\mathbb{R}$. Moreover, since h is convex, then we can rewrite (2.6), as follows

$$h\left(\frac{a+b}{2}\right)=h\left(\frac{1}{2}\left[\frac{3a+b}{4}+\frac{a+3b}{4}\right]\right)$$

$$\leq\frac{1}{2}\left[h\left(\frac{3a+b}{4}\right)+h\left(\frac{a+3b}{4}\right)\right] \tag{2.9}$$

$$\leq\int_0^1 h\big((1-t)a+bt\big)dt$$

$$\leq\frac{1}{2}\left[h\left(\frac{a+b}{2}\right)+\frac{h(a)+h(b)}{2}\right]$$

$$\leq\frac{h(a)+h(b)}{2}.$$

Let $V+iR$ be the Cartesian decomposition of $T\in\mathcal{L}(\mathcal{H})$. Then, we get

$$|T|^2+|T^*|^2=T^*T+TT^*=2(V^2+R^2)$$

and

$$\left|\langle T\hat{k}_\xi,\hat{k}_\xi\rangle\right|^2=\langle V\hat{k}_\xi,\hat{k}_\xi\rangle^2+\langle R\hat{k}_\xi,\hat{k}_\xi\rangle^2.$$

Replacing a and b with $\langle \phi(|T|^2)\hat{k}_\xi, \hat{k}_\xi\rangle$ and $\langle \phi(|T^*|^2)\hat{k}_\xi, \hat{k}_\xi\rangle$ in (2.9), for $\xi \in \Omega$, we get

$$f\left(\frac{\langle \phi(|T|^2)\hat{k}_\xi, \hat{k}_\xi\rangle + \langle \phi(|T^*|^2)\hat{k}_\xi, \hat{k}_\xi\rangle}{2}\right)$$

$$\leq \frac{1}{2}\left[f\left(\frac{3\langle \phi(|T|^2)\hat{k}_\xi,\hat{k}_\xi\rangle+\langle \phi(|T^*|^2)\hat{k}_\xi,\hat{k}_\xi\rangle}{4}\right) + f\left(\frac{\langle \phi(|T|^2)\hat{k}_\xi,\hat{k}_\xi\rangle+3\langle \phi(|T^*|^2)\hat{k}_\xi,\hat{k}_\xi\rangle}{4}\right)\right]$$

$$\leq \int_0^1 h\left((1-t)\langle \phi(|T|^2)\hat{k}_\xi, \hat{k}_\xi\rangle + \langle \phi(|T^*|^2)\hat{k}_\xi, \hat{k}_\xi\rangle\, t\right) dt$$

$$\leq \frac{1}{2}\left[f\left(\frac{\langle \phi(|T|^2)\hat{k}_\xi,\hat{k}_\xi\rangle+\langle \phi(|T^*|^2)\hat{k}_\xi,\hat{k}_\xi\rangle}{2}\right) + \frac{\left(f\langle \phi(|T|^2)\hat{k}_\xi,\hat{k}_\xi\rangle\right)+f\left(\langle \phi(|T^*|^2)\hat{k}_\xi,\hat{k}_\xi\rangle\right)}{2}\right]$$

$$\leq \frac{f\left(\langle \phi(|T|^2)\hat{k}_\xi,\hat{k}_\xi\rangle\right)+f\left(\langle \phi(|T^*|^2)\hat{k}_\xi,\hat{k}_\xi\rangle\right)}{2}.$$

However, since f is convex and ϕ is a positive unital linear map, the last two inequalities can be improved, respectively, as follows:

$$\frac{1}{2}\left[f\left(\frac{\langle \phi(|T|^2)\hat{k}_\xi, \hat{k}_\xi\rangle + \langle \phi(|T^*|^2)\hat{k}_\xi, \hat{k}_\xi\rangle}{2}\right) + \frac{\left(f\langle \phi(|T|^2)\hat{k}_\xi, \hat{k}_\xi\rangle\right) + f\left(\langle \phi(|T^*|^2)\hat{k}_\xi, \hat{k}_\xi\rangle\right)}{2}\right]$$

$$\leq \frac{1}{2}\left[f\left(\frac{\langle \phi(|T|^2)\hat{k}_\xi, \hat{k}_\xi\rangle + \langle \phi(|T^*|^2)\hat{k}_\xi, \hat{k}_\xi\rangle}{2}\right) + \frac{\left(\phi\langle f(|T|^2)\hat{k}_\xi, \hat{k}_\xi\rangle\right) + \phi\left(\langle f(|T^*|^2)\hat{k}_\xi, \hat{k}_\xi\rangle\right)}{2}\right]$$

and

$$\frac{f\left(\langle \phi(|T|^2)\hat{k}_\xi, \hat{k}_\xi\rangle\right) + f\left(\langle \phi(|T^*|^2)\hat{k}_\xi, \hat{k}_\xi\rangle\right)}{2} \leq \frac{\langle \phi\left(f(|T|^2)\right)\hat{k}_\xi, \hat{k}_\xi\rangle + \langle \phi\left(f(|T^*|^2)\right)\hat{k}_\xi, \hat{k}_\xi\rangle}{2}$$

$$= \frac{\langle \phi\left(f(|T|^2)+f(|T^*|^2)\right)\hat{k}_\xi,\hat{k}_\xi\rangle}{2}.$$

Combining the above inequalities together, we get

$$sup_{\xi\in\Omega}\int_0^1 f\left(\left\|\phi^{\frac{1}{2}}\left((1-t)|T|^2 + t|T^*|^2\right)\hat{k}_\xi\right\|^2\right) dt$$

$$\leq f\left(\left\|\phi\left(\frac{|T|^2+|T^*|^2}{2}\right)\right\|_{ber}\right) + \frac{1}{2}\phi(\|f(|T|^2) + f(|T^*|^2)\|_{ber})$$

$$\leq \frac{1}{2}\left\|\phi\left(f(|T|^2)+f(|T^*|^2)\right)\right\|_{ber}.$$

Again, since f is increasing, we obtain

$$f\left(\left|\langle\phi(|T|)\hat{k}_\xi,\hat{k}_\xi\rangle\right|^2\right)=f\left(\langle\phi(V)\hat{k}_\xi,\hat{k}_\xi\rangle^2\right)+f\left(\langle\phi(R)\hat{k}_\xi,\hat{k}_\xi\rangle^2\right)$$

$$\leq f\left(\langle\phi^2(V)\hat{k}_\xi,\hat{k}_\xi\rangle\right)+f\left(\langle\phi^2(R)\hat{k}_\xi,\hat{k}_\xi\rangle\right)$$

$$=f\left(\langle\phi(V^2+R^2)\hat{k}_\xi,\hat{k}_\xi\rangle\right)$$

$$=f\left(\frac{\langle\phi\left(|T|^2+|T^*|^2\right)\hat{k}_\xi,\hat{k}_\xi\rangle}{2}\right)$$

$$=f\left(\frac{\langle\phi\left(|T|^2\right)\hat{k}_\xi,\hat{k}_\xi\rangle+\langle\phi\left(|T^*|^2\right)\hat{k}_\xi,\hat{k}_\xi\rangle}{2}\right).$$

Taking the supremum over $\xi\in\Omega$ in all previous inequalities, we reach the required result.

Theorem 2.7: *Let* $\mathcal{H}=\mathcal{H}(\Omega)$ *be a RKHS and let* $V+iR$ *be the Cartesian decomposition of* $T\in\mathcal{L}(\mathcal{H})$. *If* $f:[0,\infty)\to[0,\infty)$ *is non-negative increasing operator convex function, then*

$$f\left(ber^2(T)\right)\geq\frac{f(\|V\|_{ber}^2)+f(\|R\|_{ber}^2)}{2}$$

$$\geq\frac{f\left(\|V^2\|_{ber}\right)+f\left(\|R^2\|_{ber}\right)}{2}$$

$$\geq\int_0^1\|tf(V^2)+(1-t)f(R^2)\|_{ber}dt \qquad (2.10)$$

$$\geq\int_0^1\|f(tV^2+(1-t)R^2)\|_{ber}dt$$

$$\geq\left\|\int_0^1 f(tV^2+(1-t)R^2)dt\right\|_{ber}$$

$$\geq\left\|f\left(\frac{T^*T+TT^*}{4}\right)\right\|_{ber}.$$

Proof: Let , $\hat{k}_\xi$ be a normalized reproducing kernel in $\mathcal{H}=\mathcal{H}(\Omega)$. Since $T=V+iR$, then we get

$$\left|\langle T\hat{k}_\xi,\hat{k}_\xi\rangle\right|^2=\langle V\hat{k}_\xi,\hat{k}_\xi\rangle^2+\langle R\hat{k}_\xi,\hat{k}_\xi\rangle^2.$$

The monotonicity of f and the above inequality show that

$$tf\left(\left|\langle T\hat{k}_\xi,\hat{k}_\xi\rangle\right|^2\right)\geq tf\left(\langle V\hat{k}_\xi,\hat{k}_\xi\rangle^2\right)$$

and

$$(1-t)f\left(\left|\langle T\hat{k}_\xi,\hat{k}_\xi\rangle\right|^2\right)\geq (1-t)f\left(\langle R\hat{k}_\xi,\hat{k}_\xi\rangle^2\right)$$

for all $t\in[0,1]$. Hence,

$$f\left(\left|\langle T\hat{k}_\xi,\hat{k}_\xi\rangle\right|^2\right)=tf\left(\left|\langle V\hat{k}_\xi,\hat{k}_\xi\rangle\right|^2\right)+(1-t)f\left(\left|\langle R\hat{k}_\xi,\hat{k}_\xi\rangle\right|^2\right)$$

$$\geq\ tf\left(\langle V\hat{k}_\xi,\hat{k}_\xi\rangle^2\right)+(1-t)f\left(\langle R\hat{k}_\xi,\hat{k}_\xi\rangle^2\right).$$

Taking the supremum over $\xi\in\Omega$, since f is increasing, we reach

$$f\left(ber^2(T)\right)\geq tf(\|V\|_{ber}^2)+(1-t)f(\|R\|_{ber}^2)$$

$$\geq tf(\|V^2\|_{ber})+(1-t)f(\|R^2\|_{ber})\ (by\ \|T^2\|_{ber}\leq\|T\|_{ber}^2, for\ all\ T\in\mathcal{L}(\mathcal{H}))$$

$$\geq t\|f(V^2)\|_{ber}+(1-t)\|f(R^2)\|_{ber}\ (by\ f(\|T\|_{ber})=\|f(|T|)\|_{ber})$$

$$\geq\ \|tf(V^2)+(1-t)f(R^2)\|_{ber}$$

$$\geq\ \|f(tV^2+(1-t)R^2)\|_{ber}\ \text{(by convexity of } f\text{)}.$$

Integrating with respect to t over $[0,1]$, we get

$$f\left(ber^2(T)\right)\geq\frac{f(\|V\|_{ber}^2)+f(\|R\|_{ber}^2)}{2}$$

$$\geq\frac{f\left(\|V^2\|_{ber}\right)+f\left(\|R^2\|_{ber}\right)}{2}$$

$$=\frac{\|f(V^2)\|_{ber}+\|f(R^2)\|_{ber}}{2}$$

$$\geq \int_0^1 \|tf(V^2) + (1-t)f(R^2)\|_{ber} dt$$

$$\geq \int_0^1 \|f(tV^2 + (1-t)R^2)\|_{ber} dt$$

$$\geq \left\| \int_0^1 f(tV^2 + (1-t)R^2) dt \right\|_{ber}$$

$$\geq \left\| f\left(\frac{T^*T+TT^*}{2}\right) \right\|_{ber} \text{(by operator convex of } f\text{)}$$

$$\geq \left\| f\left(\frac{V^2+R^2}{4}\right) \right\|_{ber}.$$

This completes the proof.

The following theorem improve inequality (2.10) and gives better estimate of Berezin radius.

Theorem 2.8: *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be an RKHS and let $V + iR$ be the Cartesian decomposition of $T \in \mathcal{L}(\mathcal{H})$. If $f: [0,\infty) \to [0,\infty)$ is non-negative increasing operator convex function, then*

$$f\left(ber^2(T)\right) \geq \frac{m}{m+n} f(\|V\|_{ber}^2) + \frac{n}{m+n} f(\|R\|_{ber}^2) \geq f\left(\frac{m\|V\|_{ber}^2 + n\|R\|_{ber}^2}{m+n}\right)$$

for all real number $m, n > 0$.

Proof: Let $\xi \in \Omega$ be an arbitrary, let $m, n > 0$ and T=$V + iR$. Then,

$$\left|\langle T\hat{k}_\xi, \hat{k}_\xi\rangle\right|^2 = \langle V\hat{k}_\xi, \hat{k}_\xi\rangle^2 + \langle R\hat{k}_\xi, \hat{k}_\xi\rangle^2.$$

The monotonicity of f and the above inequality show that

$$\frac{m}{m+n} f\left(\left|\langle T\hat{k}_\xi, \hat{k}_\xi\rangle\right|^2\right) \geq \frac{m}{m+n} f\left(\langle V\hat{k}_\xi, \hat{k}_\xi\rangle^2\right)$$

and

$$\frac{n}{m+n} f\left(\left|\langle T\hat{k}_\xi, \hat{k}_\xi\rangle\right|^2\right) \geq \frac{n}{m+n} f\left(\langle R\hat{k}_\xi, \hat{k}_\xi\rangle^2\right)$$

for all positive real number $m, n > 0$. Hence,

$$f\left(\left|\langle T\hat{k}_\xi, \hat{k}_\xi\rangle\right|^2\right) \geq \frac{m}{m+n} f\left(\langle V\hat{k}_\xi, \hat{k}_\xi\rangle^2\right) + \frac{n}{m+n} f\left(\langle R\hat{k}_\xi, \hat{k}_\xi\rangle^2\right).$$

Taking the supremum over $\in \Omega$, since f is increasing, we reach

$$\begin{aligned} f\left(ber^2(T)\right) &\geq \tfrac{m}{m+n} f(\|V\|_{ber}^2) + \tfrac{n}{m+n} f(\|R\|_{ber}^2) \\ &\geq \tfrac{m}{m+n} f(\|V^2\|_{ber}) + \tfrac{n}{m+n} f(\|R^2\|_{ber}) \ (by\ \|T^2\|_{ber} \leq \|T\|_{ber}^2, for\ all\ T \in \mathcal{L}(\mathcal{H})) \\ &\geq \tfrac{m}{m+n} \|f(V^2)\|_{ber} + \tfrac{n}{m+n} \|f(R^2)\|_{ber} \ (by\ f(\|T\|_{ber}) = \|f(|T|)\|_{ber}) \\ &\geq \left\| \tfrac{m}{m+n} f(V^2) + \tfrac{n}{m+n} f(R^2) \right\|_{ber} \\ &\geq f\left(\tfrac{m\|V\|_{ber}^2 + n\|R\|_{ber}^2}{m+n}\right). \end{aligned}$$

which obtains the desired result.

Huban et al. (Huban et al., 2022b) used the following identity

$$\left(\frac{M+N}{2}\right)^2 \leq \left(\frac{M+N}{2}\right)^2 + \left(\frac{|M-N|}{2}\right)^2 = \frac{M^2+N^2}{2} \tag{2.11}$$

for every self-adjoint operator $M, N \in \mathcal{L}(\mathcal{H})$ to prove the following improvement of the left hand side (1.3), as follows:

$$\frac{1}{4}\|T^*T + TT^*\|_{ber} \leq \frac{1}{4}\|(T^*T + TT^*)^2 + (T^2 + (T^*)^2)^2\|_{ber}^{\frac{1}{2}} \leq ber^2(T). \tag{2.12}$$

By remembering the original result in (Huban et al., 2022b), an interesting refinement to (2.12) holds. Namely, we get

$$\begin{aligned} \frac{1}{4}\|T^*T + TT^*\|_{ber} &\leq \frac{1}{4}\|(T^*T + TT^*)^2 + (T^2 + (T^*)^2)^2\|_{ber}^{\frac{1}{2}} \\ &\leq \frac{1}{4\sqrt{2}}(\|T + T^*\|_{ber}^4 + \|T - T^*\|_{ber}^4)^{1/2} \\ &\leq ber^2(T). \end{aligned} \tag{2.13}$$

The next theorem extends and improves inequality (2.13) as follows:

Theorem 2.9: *Let $V+iR$ be the Cartesian decomposition of $T\in\mathcal{L}(\mathcal{H})$. Then*

$$\frac{1}{4}\left\|\left(\frac{m-n}{m+n}\right)(T^2+(T^*)^2)+(TT^*+T^*T)\right\|_{ber}$$

$$\leq\frac{1}{4}\left\|\left[\left(\frac{m-n}{m+n}\right)(T^2+(T^*)^2)+(TT^*+T^*T)\right]^2\right.$$

$$\left.+\left[(T^2+(T^*)^2)+\left(\frac{m-n}{m+n}\right)(TT^*+T^*T)\right]^2\right\|_{ber}^{1/2}$$

$$\leq\frac{1}{2\sqrt{2}}\left(\frac{m^2\|T+T^*\|_{ber}^4+n^2\|T-T^*\|_{ber}^4}{(m+n)^2}\right)^{1/2} \tag{2.14}$$

$$\leq ber^2(T).$$

for all positive real numbers $m,n>0$.

Proof: Since $V+iR$ is the Cartesian decomposition of T, then for all real numbers $m,n>0$, we get

$$\frac{mV^2+nR^2}{m+n}=\left(\frac{m-n}{m+n}\right)\frac{(T)^2+(T^*)^2}{4}+\frac{T^*T+TT^*}{4}$$

and

$$\frac{mV^2-nR^2}{m+n}=\frac{(T)^2+(T^*)^2}{4}+\left(\frac{m-n}{m+n}\right)\frac{T^*T+TT^*}{4}.$$

Replacing M and N by $\frac{2m}{m+n}V^2$ and $\frac{2n}{m+n}R^2$ ($\forall m,n>0$), respectively, in (2.11), we get

$$\left(\frac{mV^2+nR^2}{m+n}\right)^2\leq\left(\frac{mV^2+nR^2}{m+n}\right)^2+\left(\frac{|mV^2-nR^2|}{m+n}\right)^2$$

$$=\frac{2m^2V^4+2n^2R^4}{(m+n)^2}.$$

Consequently,

$$\frac{1}{4}\left\|\left(\frac{m-n}{m+n}\right)(T^2+(T^*)^2)+(TT^*+T^*T)\right\|_{ber}^2=\left\|\frac{mV^2+nR^2}{m+n}\right\|_{ber}^2$$

$$= \left\| \left(\frac{mV^2+nR^2}{m+n} \right)^2 \right\|_{ber}$$

$$\leq \left\| \left(\frac{mV^2+nR^2}{m+n} \right)^2 + \left(\frac{|mV^2-nR^2|}{m+n} \right)^2 \right\|_{ber}$$

$$= \left\| \frac{2m^2V^4+2n^2R^4}{(m+n)^2} \right\|_{ber}$$

$$\leq \frac{2m^2\|V\|_{ber}^4+2n^2\|R\|_{ber}^4}{(m+n)^2}$$

$$\leq ber^4(T),$$

which obtains the desired result in (2.14).

Remark: *In particular, taking* $m = n$ *in (2.14), we may refer to in (2.13).*

For more recent study on Berezin radius inequalities for operators and related conclusions, we reference (Gürdal and Başaran, 2023a), (Gürdal and Başaran, 2023b), (Gürdal et al., 2023), (Huban et al., 2022b), (Garayev et al., 2023), (Bhunia et al., 2023), (Saltan et al., 2022), (Yamancı et al., 2020)).

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