Dera Natung Government College Research Journal
Volume 8 Issue 1, pp. 1-13, January-December 2023

ISSN (Print): 2456-8228
ISSN (Online): 2583-5483

Research Article

Generalized Additive Functional Equation: General Solution and Hyers-Ulam Stability in Banach Spaces via Alternative Fixed Point Theorem

Kandhasamy Tamilvanan[a], **Syed Abdul Mohiuddine**[b,c,*]

[a]Department of Mathematics, Faculty of Science & Humanities, R.M.K. Engineering College, Kavaraipettai, Tiruvallur 601 206, Tamil Nadu, India.

[b]Department of General Required Courses, Mathematics, The Applied College, King Abdulaziz University, Jeddah 21589, Saudi Arabia.

[c]Operator Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia.

Cite as: Tamilvanan, K., Mohiuddine, S.A. (2023). Generalized Additive Functional Equation: General Solution and Hyers-Ulam Stability in Banach Spaces via Alternative Fixed Point Theorem, Dera Natung Government College Research Journal, 8, 1-13. https://doi.org/10.56405/dngcrj.2023.08.01.01

Received on: 16.10.2023,

Revised on: 20.11.2023

Accepted on: 23.11.2023,

Available online: 26.12.2023.

*Corresponding Author: Syed Abdul Mohiuddine (mohiuddine@gmail.com)

Abstract: In this paper, we secure the general solution of the generalized additive functional equation

$$\varphi\left(\sum_{a=1}^{r} t_a\right) = \sum_{1\le a<b\le r} \varphi(t_a+t_b) - (r-2)\sum_{a=1}^{r} \frac{\varphi(t_a)-\varphi(-t_a)}{2}$$

where r is a positive integer with $\mathbb{N}-\{0,1,2,3,4\}$, and also examine Hyers-Ulam stability results by utilizing alternative fixed point for a generalized additive functional equation in Banach spaces.

Keywords: Banach Space, Fixed Point, Hyers-Ulam Stability, Additive Functional Equation.

MSC: 39B52, 39B72, 39B82.

I. Introduction

The study of additive functions traces its roots back to Legendre (1791), who first attempted to find solutions to the Cauchy functional equation:

$$f(x+y) = f(x) + f(y) \qquad (1.1)$$

for all $x, y \in \mathbb{R}$. A significant advancement in the investigation of the additive Cauchy functional equation was made by Cauchy (1821) in his book Cours d' Analyse in 1821. Cauchy demonstrated that every continuous additive function $f: \mathbb{R} \to \mathbb{R}$ is linear, taking the form $f(x) = mx$, where m is an arbitrary constant. Further insights into real additive functions were provided by Darboux in 1875, demonstrating that a real additive function continuous at a point is linear.

Indeed, additive functions are the solutions to the additive Cauchy functional equation (1.1). In 1985, Kuczma made significant contributions and conducted thorough research on additive functions. Furthermore, the topic of additive functions has been extensively covered in the books by Aczel (1987, 1996), highlighting their importance and relevance in the realm of mathematical analysis.

Certainly, many functional equations involving at least two factors have general solutions that can be expressed in terms of additive, multiplicative, logarithmic, or exponential functions. This versatility in functional equations is a powerful tool in mathematical analysis.

Researchers such as Eshaghi Gordji and colleagues (see (Gordji et. al, 2009), (Gordji et. al, 2010), (Gordji and Savadkouhi, 2010)) have made significant contributions to the field by proving the generalized Hyers-Ulam stability of various functional equations of mixed types, including quartic, cubic, quadratic, and additive equations, in various normed spaces. Their work has broad implications for understanding the stability of these equations.

Jung (1996) has delved into the Hyers-Ulam-Rassias stability of additive mappings and has conducted extensive investigations in Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis (Jung, 2001). This research has shed light on the stability of functional equations and their applications.

In 2005, Lee et al. specifically explored quartic functional equations, contributing to the growing body of knowledge in this field. These works collectively advance our understanding of the stability and solutions of various functional equations.

Rassias (1982, 1984, 1989, 1992) independently expressed stability of various functional equations in some normed spaces. Ulam (1960) proved and investigated functional equations stabilities in the book of “A Collection of Mathematical Problem”, which was used for many mathematicians to develop the functional equations part. Najati (2007), Najati and Park (2008), Najati and Moghimi (2008), and Najati and Rassias (2010) established additive, quadratic, cubic and quartic functional equations stability results in quasi-Banach spaces, etc. Some mathematicians were well discussed various functional equations in Banach spaces (see

(Aoki, 1950), Rassias (1978, 2000), (Tamilvanan et al., 2020, 2021, 2023)) and few articles are used to develop this paper (Czerwik, 1992), (Gajda, 1991), (Gavruta, 1996), (Hyers, 1941), (Hyers et al., 1998), (Kannappan, 1995).

In this research, we focus on establishing the general solution for a generalized additive functional equation

$$\varphi(\sum_{a=1}^{r} t_a) = \sum_{1\le a<b\le r} \varphi(t_a + t_b) - (r-2)\sum_{a=1}^{r} \frac{\varphi(t_a)-\varphi(-t_a)}{2} \qquad (1.2)$$

where r is a positive integer with $\mathbb{N} - \{0,1,2,3,4\}$, in Banach space. Moreover, we conduct a thorough examination of Hyers-Ulam stability results by employing an alternative fixed point approach for this specific type of functional equation. The exploration of both the general solution and stability aspects contributes to a comprehensive understanding of additive functional equations and their stability properties.

II. General Solution of the Functional Equation (1.2)

In this segment, we obtain the general solution of the functional equation (1.2).

Theorem 2.1. *Let N and B be real vector spaces. If the mapping $\varphi: N \to B$ fulfils the functional equation* (1.2) *for all $t_1, t_2, \dots, t_n \in N$, then $\varphi: N \to B$ fulfils the functional equation* (1.1) for all $x, y \in N$.

Proof. Let us consider $\varphi: N \to B$ fulfils the functional equation (1.2). Switching $(t_1, t_2, t_3, \dots, t_r)$ by $(0,0,0,\dots,0)$ in (1.2), we acquire $\varphi(0) = 0$. Substituting $(t_1, t_2, t_3, \dots, t_r)$ by $(t, 0,0,\dots,0)$ in (1.2), we receive $\varphi(-t) = -\varphi(t)\ \forall t \in N$. Therefore, φ is an odd function. Replacing $(t_1, t_2, t_3, \dots, t_r)$ by $(t, -t, t, -t, t, 0,0,\dots,0)$ in (1.2) and utilizing the property of odd function, we attain

$$\varphi(2t) = 2\varphi(t)\ \forall t \in N. \qquad (2.1)$$

Replacing $(t_1, t_2, t_3, \dots, t_r)$ by $(t, t, t, 0,0,\dots,0)$ in (1.2) with utilizing (2.1), we obtain

$$\varphi(3t) = 3\varphi(t)\ \forall t \in N. \qquad (2.2)$$

Switching $(t_1, t_2, t_3, \dots, t_r)$ by $(t, t, t, t, 0, \dots,0)$ in (1.2) with utilizing (2.1), we receive

$$\varphi(4t) = 4\varphi(t)\ \forall t \in N. \qquad (2.3)$$

In general, for any positive integer n, we acquire

$$\varphi(nt) = n\varphi(t)\ \forall t \in N.$$

Now, substituting $(t_1, t_2, t_3, \dots, t_r)$ by $(x, y, x, y, 0, \dots, 0)$ in (1.2), we accomplish our needed outcome (1.1).

In Segments 3 and 4, we take N be a normed space and B be a Banach space. For the notational effortlessness, we define a function $\Phi: N \to B$ by

$$\Phi(t_1, t_2, t_3, \dots, t_r) = \varphi\left(\sum_{a=1}^{r} t_a\right) - \sum_{1\le a<b\le r} \varphi(t_a + t_b) + (r-2)\sum_{a=1}^{r} \frac{\varphi(t_a) - \varphi(-t_a)}{2}$$

for every $t_1, t_2, \dots, t_r \in N$.

III. Stability Results for (1.2): Direct Method

In this segment, we examine the Hyers-Ulam stability of the generalized additive functional equation (1.2) in Banach space by means of direct method.

Theorem 3.1. *Let* $\gamma \in \{-1,1\}$ *and* $\mu: N^r \to [0,\infty)$ *be a function such that*

$$\sum_{l=0}^{\infty} \frac{\mu(2^{l\gamma} t_1, 2^{l\gamma} t_2, \dots, 2^{l\gamma} t_r)}{2^{l\gamma}}$$

converges in $\mathbb{R}$ *with*

$$\lim_{l\to\infty} \frac{\mu(2^{l\gamma} t_1, 2^{l\gamma} t_2, \dots, 2^{l\gamma} t_r)}{2^{l\gamma}} = 0\ \forall t_1, t_2, \dots, t_r \in N. \tag{3.1}$$

If $\Phi: N \to B$ *is a function fulfilling the inequality*

$$\|\Phi(t_1, t_2, \dots, t_r)\| \le \mu(t_1, t_2, \dots, t_r)\ \forall t_1, t_2, \dots, t_r \in N. \tag{3.2}$$

Then there exists a unique additive mapping $\Gamma: N \to B$ *which fulfils the functional equation* (1.2) *and*

$$\|\ \varphi(t) - \Gamma(t)\ \| \le \frac{1}{4}\sum_{l=\frac{1-\gamma}{2}}^{\infty} \frac{\mu(2^{l\gamma}t, -2^{l\gamma}t, 2^{l\gamma}t, -2^{l\gamma}t, 2^{l\gamma}t, 0, \dots, 0)}{2^{l\gamma}} \tag{3.3}$$

for all $t \in N$.

Proof. Assume that $\gamma = 1$. Switching $(t_1, t_2, \dots, t_r)$ by $(t, -t, t, -t, t, 0, \dots, 0)$ in (3.2), we have

$$\| 2\varphi(2t) - 4\varphi(t) \| \leq \mu(t, -t, t, -t, t, 0, \ldots, 0)\ \forall t \in N. \tag{3.4}$$

From (3.4), we simply get

$$\left\|\frac{\varphi(2t)}{2} - \varphi(t)\right\| \leq \frac{\mu(t,-t,t,-t,t,0,\ldots,0)}{4}\ \forall t \in N. \tag{3.5}$$

Replacing t by $2t$ in (3.5), we obtain

$$\left\|\frac{\varphi(2^2t)}{2} - \varphi(2t)\right\| \leq \frac{\mu(2t,-2t,2t,-2t,2t,0,\ldots,0)}{4} \forall \in N. \tag{3.6}$$

From (3.6), we achieve

$$\left\|\frac{\varphi(2^2t)}{2^2} - \frac{\varphi(2t)}{2}\right\| \leq \frac{1}{2}\frac{\mu(2t,-2t,2t,-2t,2t,0,\ldots,0)}{4} \forall t \in N. \tag{3.7}$$

Adding together (3.5) and (3.7), we get the following outcome

$$\left\|\frac{\varphi(2^2t)}{2^2} - \varphi(t)\right\| \leq \frac{1}{4}\left[\mu(t, -t, t, -t, t, 0, \ldots, 0) + \frac{\mu(2t,-2t,2t,-2t,2t,0,\ldots,0)}{2}\right]\ \forall t \in N. \tag{3.8}$$

It follows from (3.5), (3.7) and (3.8), we can generalizing that as follows

$$\begin{aligned} \left\|\frac{\varphi(2^rt)}{2^r} - \varphi(t)\right\| &\leq \frac{1}{4}\sum_{l=0}^{r-1} \frac{\mu(2^lt,-2^lt,2^lt,-2^lt,2^lt,0,\ldots,0)}{2^l}\ \forall t \in N \\ \left\|\frac{\varphi(2^rt)}{2^r} - \varphi(t)\right\| &\leq \frac{1}{4}\sum_{l=0}^{\infty} \frac{\mu(2^lt,-2^lt,2^lt,-2^lt,2^lt,0,\ldots,0)}{2^l}\ \forall t \in N. \end{aligned} \tag{3.9}$$

In order to establish convergence of the sequence $\left\{\frac{\varphi(2^lt)}{2^l}\right\}$, replacing t by 2^mt also dividing by 2^m in (3.9). We then deduce that, for any $l, m > 0$,

$$\begin{aligned} \left\|\frac{\varphi(2^{l+m}t)}{2^{(l+m)}} - \frac{\varphi(2^mt)}{2^m}\right\| &= \frac{1}{2^m}\left\|\frac{\varphi(2^{l+m}t)}{2^l} - \varphi(2^lt)\right\| \\ &\leq \frac{1}{4}\sum_{l=0}^{r-1} \frac{\mu(2^{l+m}t,-2^{l+m}t,2^{l+m}t,-2^{l+m}t,2^{l+m}t,0,\ldots,0)}{2^{(l+m)}} \\ &\leq \frac{1}{4}\sum_{l=0}^{\infty} \frac{\mu(2^{l+m}t,-2^{l+m}t,2^{l+m}t,-2^{l+m}t,2^{l+m}t,0,\ldots,0)}{2^{(l+m)}} \\ &\to 0 \text{ as } m \to \infty \end{aligned} \tag{3.10}$$

for all $t \in N$. Therefore, the sequence $\left\{\frac{\varphi(2^lt)}{2^l}\right\}$ is a Cauchy sequence. As B is complete, there exists a mapping $\Gamma: N \to B$ such that $\Gamma(t) = \lim_{l\to\infty} \frac{\varphi(2^lt)}{2^l}\ \forall t \in N$. Taking $l \to \infty$ in (3.9), we attain the outcome that (3.3) holds for all $t \in N$. To confirm that Γ fulfils (1.2), replacing $(t_1, t_2, \ldots, t_r)$ by $(2^l t_1, 2^l t_2, \ldots, 2^l t_r)$ also dividing by 2^l in (3.2), we attain

$$\frac{1}{2^l}\|\Phi(2^l t_1, 2^l t_2, \ldots, 2^l t_r)\| \leq \frac{1}{2^l}\mu(2^l t_1, 2^l t_2, \ldots, 2^l t_r)\ \forall t_1, t_2, \ldots, t_r \in N.$$

Letting $l \to \infty$ in the above inequality with utilizing the definition of $\Gamma(t)$, we observe that $\Gamma(t_1, t_2, \ldots, t_r) = 0\ \forall t_1, t_2, \ldots, t_r \in N$. Thus Γ fulfils (1.2). To illustrate that Γ is unique, let $\Omega(t)$ be an another additive mapping fulfilling (1.2) also (3.3). Then

$$\begin{aligned}\| \Gamma(t) - \Omega(t) \| &\leq \frac{1}{2^m}\|\Gamma(2^m t) - \varphi(2^m t)\| + \|\varphi(2^m t) - \Omega(2^m t)\| \\ &\leq \frac{1}{4}\sum_{l=0}^{\infty} \frac{\mu(2^{l+m}t, -2^{l+m}t, 2^{l+m}t, -2^{l+m}t, 2^{l+m}t, 0, \ldots, 0)}{2^{(l+m)}} \\ &\to 0 \text{ as } m \to \infty\end{aligned}$$

for all $t \in N$. Therefore, Γ is unique. Now, switching t by $\frac{t}{2}$ in (3.4), we have

$$\left\|2\varphi(t) - 4\varphi\left(\frac{t}{2}\right)\right\| \leq \mu\left(\frac{t}{2}, -\frac{t}{2}, \frac{t}{2}, -\frac{t}{2}, \frac{t}{2}, 0, \ldots, 0\right)\ \forall t \in N. \tag{3.11}$$

The rest of the proof is similar to that when $\gamma = 1$. So, for $\gamma = -1$, we can demonstrate the consequence through homogeneous procedure. This accomplished the proof of the theorem.

The upcoming corollary is an instantaneous outcome of Theorem 3.1 regarding the stability for the equation (1.2).

Corollary 3.2. *Let α and β be positive real numbers. If there exists a function $\Phi: N \to B$ fulfilling the inequality*

$$\|\Phi(t_1, t_2, \ldots, t_r)\| \leq \begin{cases} \alpha \\ \alpha\left\{\sum_{a=1}^{r} \|t_a\|^{\beta}\right\} \\ \alpha\left\{\Pi_{a=1}^{r}\|t_a\|^{\beta} + \sum_{a=1}^{r} \|t_a\|^{r\beta}\right\} \end{cases} \tag{3.12}$$

for all $t_1, t_2, \ldots, t_r \in N$, then there exists a unique additive function $\Gamma: N \to B$ such that

$$\| \varphi(t) - \Gamma(t) \| \leq \begin{cases} \frac{\alpha}{|2|} \\ \frac{\alpha\|t\|^{\beta}}{2|2-2^{\beta}|}; \quad \beta \neq 1 \\ \frac{\alpha|t\|^{r\beta}}{2|2-2^{r\beta}}; \quad \beta \neq \frac{1}{r} \end{cases} \tag{3.13}$$

for all $t \in N$.

IV. Stability Results for (1.2) : Fixed Point Method

In this segment, we scrutinize the generalized Hyers-Ulam stability of the generalized additive functional equation (1.2) in Banach space through fixed point method.

Theorem 4.1. *Let* $\Gamma: N \rightarrow B$ *be a mapping for which there exists a function* $\mu: N^r \rightarrow [0,\infty)$ *and*

$$\lim_{l\to\infty} \frac{\mu(\phi_a^l t_1, \phi_a^l t_2, \dots, \phi_a^l t_r)}{\phi_a^l} = 0, \tag{4.1}$$

where $\phi_a = \begin{cases} 2 \; if \; a=0 \\ \frac{1}{2} \; if \; a=1 \end{cases}$ *and such that it fulfils the inequality*

$$\|\Gamma(t_1, t_2, \dots, t_r)\| \le \mu(t_1, t_2, \dots, t_r) \; \forall t_1, t_2, \dots, t_r \in N. \tag{4.2}$$

If there exists $L = L(a)$ *such that*

$$t \rightarrow \sigma(t) = \frac{1}{2}\mu\left(\frac{t}{2}, -\frac{t}{2}, \frac{t}{2}, -\frac{t}{2}, \frac{t}{2}, 0, \dots, 0\right)$$

has the property

$$\frac{\sigma(\phi_a t)}{\phi_a} = L\sigma(t) \; \forall t \in N. \tag{4.3}$$

Then there exists a unique additive function $\Gamma: N \rightarrow B$ *fulfilling* (1.2) *and such that*

$$\| \varphi(t) - \Gamma(t) \| \le \frac{L^{1-a}}{1-L}\sigma(t) \; \forall t \in N. \tag{4.4}$$

Proof. Choose the set $\chi = \{y/y: N \rightarrow B, y(0) = 0\}$ and initiate the generalized metric on χ, $d(y,z) = \inf\{l \in (0,\infty): \| y(t) - z(t) \| \le l\sigma(t), t \in \Gamma\}$. It is simple to observe that (χ, d) is complete. Define $\eta: \chi \rightarrow \chi$ by $\eta y(t) = \frac{1}{\phi_a} y(\phi_a t) \; \forall t \in N$. For $y, z \in \chi$ and $t \in N$, we have that

$$d(y,z) = l \Rightarrow \| y(t) - z(t) \| \le l\sigma(t) \Rightarrow \left\| \frac{y(\phi_a t)}{\phi_a} - \frac{z(\phi_a t)}{\phi_a} \right\| \le \frac{1}{\phi_a} l\sigma(\phi_a t)$$
$$\Rightarrow \| \eta y(t) - \eta z(t) \| \le \frac{1}{\phi_a} l\sigma(\phi_a t) \Rightarrow d(\eta y(t), \eta z(t)) \le lL$$

That is, $d(\eta y, \eta z) \leq Ld(y,z)$. Accordingly, η is a strictly contractive mapping on χ with Lipschitz constant L. It is follows from (3.4) that

$$\| 2\varphi(2t) - 4\varphi(t) \| \leq \mu(t,-t,t,-t,t,0,\dots,0)\ \forall t \in N. \tag{4.5}$$

It is follows from (4.5) that

$$\left\|\frac{\varphi(2t)}{2} - \varphi(t)\right\| \leq \frac{\mu(t,-t,t,-t,t,0,\dots,0)}{4} \forall t \in N. \tag{4.6}$$

Utilizing (4.3) for the case $a = 0$, our outcome is reduced to

$$\left\|\varphi(t) - \frac{\varphi(2t)}{2}\right\| \leq \frac{1}{2}\sigma(t) \Rightarrow \| \varphi(t) - \eta y(t) \| \leq L\sigma(t)\ \forall t \in N.$$

So, we obtain the outcome that

$$d\big(\eta\varphi(t), \varphi(t)\big) \leq L = L^{1-a} < \infty\ \forall t \in N. \tag{4.7}$$

Replacing t by $\frac{t}{2}$ in (4.6), we have

$$\left\|\frac{\varphi(t)}{2} - \varphi\left(\frac{t}{2}\right)\right\| \leq \frac{\mu\left(\frac{t}{2},-\frac{t}{2},\frac{t}{2},-\frac{t}{2},\frac{t}{2},0,\dots,0\right)}{4}\ \forall t \in N. \tag{4.8}$$

Utilizing (4.3) for the case $a = 1$, our outcome is reduced to

$$\left\|2\varphi\left(\frac{t}{2}\right) - \varphi(t)\right\| \leq \sigma(t) \Rightarrow \| \eta\varphi(t) - \varphi(t) \| \leq \sigma(t)\ \forall t \in N.$$

Therefore, we acquire the outcome that

$$d\big(\varphi(t), \eta\varphi(t)\big) \leq \frac{1}{2} = L^{1-a}\ \forall t \in N. \tag{4.9}$$

Utilizing (4.7) with (4.9), we can conclude

$$d\big(\varphi(t), \eta y(t)\big) \leq L^{1-a} < \infty\ \forall t \in N. \tag{4.10}$$

Now from the fixed point alternative theorem, it follows that there exists a fixed point Γ of η in χ such that

$$\Gamma(t) = \lim_{l\to\infty} \frac{\varphi(\phi_a^l t)}{\phi_a^l} \forall t \in N. \tag{4.11}$$

In order to establish $\Gamma: N \to B$ fulfils (1.2), we use an argument similar to that in the proof of Theorem 3.1. As Γ is a unique fixed point of η in the set $\Delta = \{\varphi \in \chi / d(\varphi, \Gamma) < \infty\}$, Γ is a unique function such that

$$d(\varphi, \Gamma) \leq \frac{1}{1-L} d(\varphi, \eta\varphi) \Rightarrow d(\varphi, \Gamma) \leq \frac{L^{1-a}}{1-L} \Rightarrow \| \varphi(t) - \Gamma(t) \| \leq \frac{L^{1-a}}{1-L} \sigma(t)$$

for all $t \in N$. This accomplished the proof of the Theorem.

The upcoming corollary is an instantaneous outcome of Theorem 4.1, regarding the stability for the equation (1.2).

Corollary 4.2. *Let α and β be positive real numbers. If a function $\Phi: N \to B$ fulfils the inequality* (3.12) *for all $t_1, t_2, \dots, t_r \in N$, then there exists a unique additive function such that* (3.13) *for all $t \in N$.*

Proof. We set

$$\mu(t_1, t_2, \dots, t_r) \leq \begin{cases} \alpha \\ \alpha \left\{ \sum_{a=1}^{r} \|t_a\|^{\beta} \right\} \\ \alpha \left\{ \Pi_{a=1}^{r} \|t_a\|^{\beta} + \sum_{a=1}^{r} \|t_a\|^{r\beta} \right\} \end{cases}$$

for all $t_1, t_2, \dots, t_r \in N$. Now

$$\frac{\mu(\phi_a^l t_1, \phi_a^l t_2, \dots, \phi_a^l t_r)}{\phi_a^l} = \begin{cases} \frac{\alpha}{\phi_a^l}, \\ \frac{\alpha}{\phi_a^l} \left\{ \sum_{a=1}^{r} \|\phi_a t_a\|^{\beta} \right\} \\ \frac{\alpha}{\phi_a^l} \left\{ \Pi_{a=1}^{r} \|\phi_a t_a\|^{\beta} + \sum_{a=1}^{r} \|\phi_a t_a\|^{r\beta} \right\} \end{cases} = \begin{cases} \to 0 \ \text{as} \ l \to \infty \\ \to 0 \ \text{as} \ l \to \infty \\ \to 0 \ \text{as} \ l \to \infty \end{cases}$$

i.e., (4.1) holds. As we have

$$\sigma(t) = \frac{1}{2} \mu \left(\frac{t}{2}, -\frac{t}{2}, \frac{t}{2}, -\frac{t}{2}, \frac{t}{2}, 0, \dots, 0 \right)$$

then

$$\sigma(t) = \frac{1}{2}\mu\left(\frac{t}{2}, -\frac{t}{2}, \frac{t}{2}, -\frac{t}{2}, \frac{t}{2}, 0, \dots, 0\right) = \begin{cases} \frac{\alpha}{2}, \\ \frac{\alpha \parallel t \parallel^{\beta}}{2(2^{\beta})} \\ \frac{\alpha \parallel t \parallel^{r\beta}}{2(2^{r\beta})}. \end{cases},$$

Also,

$$\frac{1}{\phi_a}\sigma(\phi_a t) = \begin{cases} \frac{\alpha}{2\phi_a}, \\ \frac{\alpha}{2\phi_a}\frac{\alpha \parallel t \parallel^{\beta} \phi_a^{\beta}}{2^{\beta}}, \\ \frac{\alpha}{2\phi_a}\frac{\alpha \parallel t \parallel^{r\beta} \phi_a^{r\beta}}{2^{r\beta}}, \end{cases} = \begin{cases} \phi_a^{-1}\sigma(t) \\ \phi_a^{\beta-1}\sigma(t) \\ \phi_a^{r\beta-1}\sigma(t) \end{cases}$$

for all $t \in N$. As the inequality (1.2) holds for the following cases:

$L = 2^{-1}$ if $a = 0$ and $L = 2^{1}$ if $a = 1$;

$L = 2^{\beta-1}$ for $\beta < 1$ if $a = 0$ and $L = 2^{1-\beta}$ for $\beta > 1$ if $a = 1$;

$L = 2^{r\beta-1}$ for $\beta < \frac{1}{r}$ if $a = 0$ and $L = 2^{1-r\beta}$ for $\beta > \frac{1}{r}$ if $a = 1$.

Now from (4.4), we verify the following cases:

Case 1: $L = 2^{-1}$ if $a = 0$.

$$\parallel \varphi(t) - \Gamma(t) \parallel \leq \frac{L^{1-a}}{1-L}\sigma(t) = \frac{2^{-1}\alpha}{2(1-2^{-1})} = \frac{\alpha}{2}.$$

Case 2: $L = 2$ if $a = 1$.

$$\parallel \varphi(t) - \Gamma(t) \parallel \leq \frac{L^{1-a}}{1-L}\sigma(t) = \frac{\alpha}{2(1-2)} = -\frac{\alpha}{2}.$$

Case 3: $L = 2^{\beta-1}$ for $\beta < 1$ if $a = 0$.

$$\parallel \varphi(t) - \Gamma(t) \parallel \leq \frac{L^{1-a}}{1-L}\sigma(t) = \frac{2^{\beta-1}}{1-2^{\beta-1}}\frac{\alpha\|t\|^{\beta}}{2(2^{\beta})} = \frac{\alpha\|t\|^{\beta}}{2(2-2^{\beta})}.$$

Case 4: $L = 2^{1-\beta}$ for $\beta > 1$ if $a = 1$.

$$\| \varphi(t) - \Gamma(t) \| \leq \frac{L^{1-a}}{1-L} \sigma(t) = \frac{1}{1-2^{1-\beta}} \frac{\alpha \|t\|^{\beta}}{2(2^{\beta})} = \frac{\alpha \|t\|^{\beta}}{2(2^{\beta}-2)}.$$

Case 5: $L = 2^{r\beta-}$ for $\beta < \frac{1}{r}$ if $a = 0$.

$$\| \varphi(t) - \Gamma(t) \| \leq \frac{L^{1-a}}{1-L} \sigma(t) = \frac{2^{r\beta-1}}{1-2^{r\beta-1}} \frac{\alpha \|t\|^{r\beta}}{2(2^{r\beta})} = \frac{\alpha \|t\|^{r\beta}}{2(2-2^{r\beta})}.$$

Case 6: $L = 2^{1-r\beta}$ for $\beta > \frac{1}{r}$ if $a = 1$.

$$\| \varphi(t) - \Gamma(t) \| \leq \frac{L^{1-a}}{1-L} \sigma(t) = \frac{1}{1-2^{1-r\beta}} \frac{\alpha \|t\|^{r\beta}}{2(2^{r\beta})} = \frac{\alpha \|t\|^{r\beta}}{2(2^{r\beta}-2)}.$$

Hence the proof is accomplished.

Availability of Data and Materials: No data were used to support the findings of the study.

Conflicts of Interest: The authors declare that they have no conflict interests.

Funding: There was no funding support for this study.

Authors' Contributions: All authors contributed equally to this work. And all the authors have read and approved the final version manuscript.

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