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Research Article

On Ideal Convergence in Generalized Metric Spaces

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Abstract: Our main goal in this paper is to introduce the concept of ideal convergence in G-metric spaces. We give definitions of $G\mathcal{I}$-convergence and $G\mathcal{I}^*$-convergence in G-metric spaces. We also extend the $\mathcal{I}$-convergence concept's properties to $G\mathcal{I}$-convergence. Then we demonstrate that $G\mathcal{I}$-convergence and $G\mathcal{I}^*$-convergence are equivalent by giving the property (AP) definition. Additionally we introduce $G\mathcal{I}$-Cauchy and $G\mathcal{I}^*$-Cauchy sequences and adapt the classically stated theorems to G-metric spaces.

Keywords: G-metric spaces, $G\mathcal{I}$-convergence, $G\mathcal{I}^*$-convergence, $G\mathcal{I}$-Cauchy sequence, $G\mathcal{I}^*$-Cauchy sequence.

I. Introduction and Preliminaries

Fast (Fast, 1951) and Steinhaus' (Steinhaus, 1951) introduction of the idea of statistical convergence resulted in the development of a new area of mathematics research and the conduct of numerous studies. Considering Fridy's paper (Fridy, 1985), the natural density of a subset A of $\mathbb{N}$, which inspired the concept of statistical convergence, defined as

$$\delta(A) = \lim_n \frac{|A_n|}{n} = \lim_n \frac{1}{n} |\{k \in A : k \leq n\}|.$$

Remember the idea of statistical convergence in the context of this definition:

A sequence (u_k) of real numbers is said to be statistically convergent to u provided that for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : |u_k - u| \geq \varepsilon\}| = 0.$$

This is denoted by $st - lim\, u_k = u$.

The concept of ideal convergence, which provides a unifying perspective on various types of convergence related to the concept of statistical convergence, was defined by Kostyrko et al. (Kostyrko, Salat, & Wilczynski, 2000). Various studies have conducted with the help of ideal convergence and other types of convergence (see (Demirci, Kişi, & Gürdal, 2023), (Gürdal & Kişi, 2022), (Nabiev, Savaş, & Gürdal, 2019), (Savaş & Gürdal, 2015), (Yamancı & Gürdal, 2013), (Şahiner, Gürdal, & Yiğit, 2011)).

Before introducing $\mathcal{I}$-convergence, we first discuss the concepts of ideal and filter, followed by proposition, which describes the relationship between them.

Definition 1.1: Let X is a non-empty set. A family $\mathcal{I} \subseteq 2^X$ of subsets of X is said to be an ideal if following conditions are provided:

(i) $\emptyset \in \mathcal{I}$,
(ii) $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
(iii) $A \in \mathcal{I}, B \subseteq A \Rightarrow B \in \mathcal{I}$.

An ideal is called non-trivial if $X \notin \mathcal{I}$. Also, a non-trivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Definition 1.2: Let X is a non-empty set. A non-empty family $\mathcal{F} \subseteq 2^X$ of subsets of X is said to be a filter if following conditions are provided:

(i) $\emptyset \notin \mathcal{F}$,
(ii) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$,
(iii) $A \in \mathcal{F}, A \subseteq B \Rightarrow B \in \mathcal{F}$.

Proposition 1.3: Let X is a non-empty set and $\mathcal{I}$ be a non-trivial ideal in X. Then the class

$$\mathcal{F}(\mathcal{I}) = \{M \subseteq X : \exists A \in \mathcal{I}, M = X \setminus A\}$$

is a filter on X and the notation $\mathcal{F}(\mathcal{I})$ is called the filter associated with $\mathcal{I}$.

Now we introduce what $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence, $\mathcal{I}$-Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence mean. The following definitions let (X, d) be a metric space and $\mathcal{I}$ be a non-trivial ideal of subsets of $\mathbb{N}$.

Definition 1.4 **(Kostyrko, Salat, & Wilczynski, 2000)*:*** A sequence (u_n) of elements of X is said to be $\mathcal{I}$-convergent to $u \in X$, if for every $\varepsilon > 0$,

$$A(\varepsilon) = \{n \in \mathbb{N}: d(u_n, u) \geq \varepsilon\} \in \mathcal{I}.$$

The element u is called the $\mathcal{I}$-limit of the sequence (u_n) and the notation $\mathcal{I} - lim u_n = u$ is used.

The theory of statistical convergence is well known for the following result:

A sequence (u_n) of real numbers is statistically convergent to u if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $\delta(M) = 1$ and $lim u_{m_k} = u$. This result suggests the introduction of a new concept of convergence, $\mathcal{I}^*$-convergence, which is closely related to $\mathcal{I}$-convergence.

Definition 1.5 **(Kostyrko, Salat, & Wilczynski, 2000)*:*** A sequence (u_n) of elements of X is said to be $\mathcal{I}^*$-convergent to $u \in X$, if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}, M \in \mathcal{F}(\mathcal{I})$ such that

$$\lim_k d(u_{m_k}, u) = 0.$$

The element u is called the $\mathcal{I}^*$-limit of the sequence (u_n) and the notation $\mathcal{I}^* - lim u_n = u$ is used.

Definition 1.6 **(Nabiev, Pehlivan, & Gürdal, 2007)*:*** An admissible ideal $\mathcal{I}$ is said to satisfy the property (AP), if for every countable family of mutually disjoint sets $\{A_1, A_2, \cdots\}$ belongs to $\mathcal{I}$ there exists countable family of sets $\{B_1, B_2, \cdots\}$ such that $A_j \triangle B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

This definition is important for explaining the relationship between $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence.

The following definitions let (X, d) be a metric space and $\mathcal{I}$ be an admissible ideal of subsets of $\mathbb{N}$.

Definition 1.7 **(Nabiev, Pehlivan, & Gürdal, 2007)*:*** A sequence (u_n) of elements of X is said to be $\mathcal{I}$-Cauchy sequence in X, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$A(\varepsilon) = \{n \in \mathbb{N}: d(u_n, u_{n_0}) \geq \varepsilon\} \in \mathcal{I}.$$

Definition 1.8 **(Nabiev, Pehlivan, & Gürdal, 2007)*:*** A sequence (u_n) of elements of X is said to be $\mathcal{I}^*$-Cauchy sequence in X, if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}, M \in \mathcal{F}(\mathcal{I})$ such that the subsequence $u_M = (u_{m_k})$ is an ordinary Cauchy sequence in X, i.e.,

$$\lim_{k,p\to\infty} d\left(u_{m_k}, u_{m_p}\right) = 0.$$

Studies on generalizing metric spaces were first carried out by Gahler (Gahler, 1963), (Gahler, 1966) and Dhage (Dhage, 1992). As a result of these studies, the inadequacy of some axioms was eliminated, and a correct generalization of the classically known concept of metric was given by Mustafa and Sims (Mustafa & Sims, 2003), (Mustafa & Sims, 2006). With the help of the work of Mustafa and Sims, Choi et al. (Choi, Kim, & Yang, 2018) presented a more general definition, called the g-metric, which measures the distance between n-points and then Abazari (Abazari, 2022) defined statistical convergence in g-metric spaces. New investigations on the concept of generalized metric space and statistical convergence have developed as a result of these studies (Gürdal, Kolancı, & Kişi, 2023), (Gürdal, Kişi, & Kolancı, 2023), (Kolancı, Gürdal, & Kişi, 2023).

Now we first introduce G-metric spaces, and then, using Abazari's definition, we redefine the concept of statistical convergence in this space, which is the main part of our study and will be useful to us in the following sections.

Definition 1.9 **(Mustafa & Sims, 2006)*:*** Let X be a nonempty set and $G: X \times X \times X \to \mathbb{R}^+$ be a function satisfying:

(i) $G(u,v,w) = 0$ if $u = v = w$,
(ii) $G(u,u,v) > 0$ for all $u, v \in X$ with $u \neq v$,
(iii) $G(u,u,v) \leq G(u,v,w)$ for all $u, v, w \in X$ with $v \neq w$,
(iv) $G(u,v,w) = G(v,u,w) = \cdots = G(w,v,u)$ (symmetry in all three variables),
(v) $G(u,v,w) \leq G(u,a,a) + G(a,v,w)$ for all $u, v, w, a \in X$ (rectangle inequality).

The function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X, G) is a G-metric space.

Proposition 1.10 **(Mustafa & Sims, 2006)*:*** Let (X, G) be a G-metric space, then for all $u, v, w, a \in X$ it follows that:

(i) If $G(u,v,w) = 0$, then $u = v = w$,
(ii) $G(u,v,w) \leq G(u,u,v) + G(u,u,w)$,
(iii) $G(u,v,v) \leq 2G(v,u,u)$,
(iv) $G(u,v,w) \leq G(u,a,w) + G(a,v,w)$,
(v) $G(u,v,w) \leq \frac{2}{3}[G(u,v,a) + G(u,a,w) + G(a,v,w)]$,

(vi) $G(u,v,w) \leq [G(u,a,a) + G(v,a,a) + G(w,a,a)]$.

Definition 1.11 **(Mustafa & Sims, 2006)*:*** Let (X,G) be a G-metric space, $u \in X$ be a point and (u_n) be a sequence in X.

(i) (u_n) G-converges to u, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that
$$G(u,u_j,u_k) < \varepsilon$$
for all $j,k \geq n_0$. The element u is called the G-limit of the sequence (u_n) and the notation $G - lim\, u_n = u$ is used.

(ii) (u_n) is said to be G-Cauchy sequence, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that
$$G(u_j,u_k,u_l) < \varepsilon$$
for all $j,k,l \geq n_0$.

To define statistical convergence in G-metric spaces, we utilize the work of (Abazari, 2022).

Definition 1.12: Let $A \in \mathbb{N}^2$ and $A(n) = \{(j,k) \in A: j,k \leq n\}$, then

$$\delta_2(A) = \lim_{n} \frac{2}{n^2} |A(n)|$$

is called 2-dimensional natural density of the set A.

Definition 1.13: Let (u_n) be a sequence in a G-metric space (X,G).

(i) (u_n) is said to be G-statistically convergent to u, if for every $\varepsilon > 0$,
$$\lim_{n} \frac{2}{n^2} \left|\{(j,k) \in \mathbb{N}^2: j,k \leq n, G(u,u_j,u_k) \geq \varepsilon\}\right| = 0$$
and is denoted by $GS - limu_n = u$.

(ii) (u_n) is said to be statistically G-Cauchy, if for every $\varepsilon > 0$ and there exist $n_0 \in \mathbb{N}$ such that
$$\lim_{n} \frac{2}{n^2} \left|\{(j,k) \in \mathbb{N}^2: j,k \leq n, G(u_{n_0},u_j,u_k) \geq \varepsilon\}\right| = 0.$$

II. Main Results

In the following (X,G) be a G-metric space and $\mathcal{I}_2$ denotes a non-trivial ideal of subsets of $\mathbb{N}^2$ where $A \in \mathcal{I}_2$ such that $A = \{(j,k): j,k \in \mathbb{N}\}$.

Definition 2.1: A sequence (u_n) of elements of X is said to be $G\mathcal{J}$-convergent to $u \in X$, if for every $\varepsilon > 0$,

$$A_G(\varepsilon) = \{(j,k) \in \mathbb{N}^2 : G(u, u_j, u_k) \geq \varepsilon\} \in \mathcal{J}_2.$$

The element u is called the $G\mathcal{J}$-limit of the sequence (u_n) and the notation $G\mathcal{J} - lim\, u_n = u$ is used.

Remark 2.2: Let $\mathcal{J}_2$ be a non-trivial ideal of subsets of $\mathbb{N}^2$. The following are examples of admissible ideals:

$$\mathcal{J}_f = \{A \subset \mathbb{N}^2 : A \text{ is finite}\}$$
$$\mathcal{J}_{\delta_2} = \{A \subset \mathbb{N}^2 : \delta_2(A) = 0\}.$$

Let we define the set of A as

$$A_G(\varepsilon) = \{(j,k) \in \mathbb{N}^2 : G(u, u_j, u_k) \geq \varepsilon\}.$$

If the $\mathcal{J}_f$ ideal is chosen instead of the $\mathcal{J}_2$ ideal, then $G\mathcal{J}$-convergence and G-convergence are equivalent. In the same way, if the $\mathcal{J}_{\delta_2}$ ideal is chosen instead of the $\mathcal{J}_2$ ideal, then $G\mathcal{J}$-convergence and statistical convergence in G-metric spaces are equivalent.

Proposition 2.3: Let (u_n) be a sequence in X and $\mathcal{J}_2$ is an admissible ideal. If the sequence (u_n) G-converges to u, then this sequence is $G\mathcal{J}$-convergent to u.

Proof: Suppose that the sequence (u_n) is G-convergent to u in (X, G). Then, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $j, k \geq n_0$,

$$G(u, u_j, u_k) < \varepsilon.$$

Thus, we have that the set of

$$A_G(\varepsilon) = \{(j,k) \in \mathbb{N}^2 : G(u, u_j, u_k) \geq \varepsilon\}$$

is included by a finite set B dependent on n_0. Since $\mathcal{J}_2$ is an admissible ideal and $\{B \subset \mathbb{N}^2 : B \text{ is finite}\} \subset \mathcal{J}_2$, we get that the set of B belongs to $\mathcal{J}_2$. So $A_G(\varepsilon) \in \mathcal{J}_2$, i.e., $G\mathcal{J} - lim\, u_n = u$.

The well-known convergence axioms of classical convergence are listed below. Now let us examine whether $G\mathcal{J}$-convergence satisfies:

a. Every constant sequence $(u_n) = (u, u, \dots, u, \dots)$ converges to u.

b. The limit of any convergent sequence is determined uniquely.
c. If a sequence (u_n) has the limit u, then each of its subsequences has the same limit.
d. If each subsequence of the sequence (u_n) has a subsequence which converges to u, then (u_n) converges to u.

Theorem 2.4: Let $\mathcal{J}_2$ be an admissible ideal. Then,

(i) $G\mathcal{J}$-convergence satisfies the axioms (a), (b) and (d).
(ii) If $\mathcal{J}_2$ contains an infinite set, then $G\mathcal{J}$-convergence does not satisfy (c).

Proof: (i) As $\mathcal{J}_2$ is an admissible ideal, it is obvious that $G\mathcal{J}$-convergence satisfies the axiom (a).

Suppose that (u_n) be a sequence in X, $G\mathcal{J} - lim\, u_n = u$, $G\mathcal{J} - lim\, u_n = v$ and $u \neq v$. For arbitrary $\varepsilon > 0$,

$$A_G(\varepsilon) = \left\{(j,k) \in \mathbb{N}^2 : G(u, u_j, u_k) \geq \frac{\varepsilon}{4}\right\},$$

$$B_G(\varepsilon) = \left\{(j,k) \in \mathbb{N}^2 : G(v, u_j, u_k) \geq \frac{\varepsilon}{4}\right\}.$$

Since $A_G(\varepsilon), B_G(\varepsilon) \in \mathcal{J}_2$, the sets $(\mathbb{N}^2 \setminus A_G(\varepsilon))$ and $(\mathbb{N}^2 \setminus B_G(\varepsilon))$ belong to the filter $\mathcal{F}(\mathcal{J}_2)$. By the properties of filter $\emptyset \neq (\mathbb{N}^2 \setminus A_G(\varepsilon)) \cap (\mathbb{N}^2 \setminus B_G(\varepsilon)) \in \mathcal{F}(\mathcal{J}_2)$. There exists a element $(j,k) \in (\mathbb{N}^2 \setminus A_G(\varepsilon)) \cap (\mathbb{N}^2 \setminus B_G(\varepsilon))$ such that

$$\begin{aligned} G(u,v,v) &\leq G(u,u_j,u_j) + G(u_j,v,v) \\ &\leq G(u,u_j,u_j) + 2G(v,u_j,u_j) \\ &\leq 2[G(u,u_j,u_k) + G(v,u_j,u_k)] \\ &< 2\left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) = \varepsilon \end{aligned}$$

Thus, we have $u = v$ and $G\mathcal{J}$-convergence satisfies the axiom (b).

To demonstrate that axiom (d) is true, we can show that the following statement is equivalent to (d):

If (u_n) is not $G\mathcal{J}$-convergent to u, then there exists a subsequence (w_k) of (u_n) such that no subsequence of (w_k) is $G\mathcal{J}$-convergent to u. By the definition of $G\mathcal{J}$-convergence, $\varepsilon_0 > 0$ such that

$$A_G(\varepsilon_0) = \{(j,k) \in \mathbb{N}^2 : G(u, u_j, u_k) \geq \varepsilon\} \notin \mathcal{I}_2. \tag{1}$$

In that case $A_G(\varepsilon_0)$ is an infitine set since $\mathcal{I}_2$ is admissible. We can choose (n_k) as an increasing sequence in $\mathbb{N}$. Put $(w_k) = (u_{n_k})$ for $k \in \mathbb{N}$ and $w_k = u_{n_k}$ for $n_k < k \leq n_{k+1}$, $j, k \leq n_{k+1}$ is a subsequence of (u_n) and by (1).

$$\{(j,k) \in \mathbb{N}^2 : G(u, u_j, u_k) \geq \varepsilon\} \notin \mathcal{I}_2. \tag{2}$$

From (2) we can see that there is no subsequence of (w_k) can be $G\mathcal{I}$-convergent to u. Thus the theorem is proved.

Numerous arithmetical properties of $G\mathcal{I}$-convergence are similar to those of usual convergence.

Theorem 2.5: Let $\mathcal{I}_2$ be a non-trivial ideal.

(i) If $G\mathcal{I} - lim\, u_n = u$ and $G\mathcal{I} - lim\, v_n = v$, then $G\mathcal{I} - lim\, (u_n + v_n) = u + v$.
(ii) If $G\mathcal{I} - \lim u_n = u$ and $G\mathcal{I} - \lim v_n = v$, then $G\mathcal{I} - \lim(u_n v_n) = uv$.

Proof: (i) Let $\varepsilon > 0$. Since $G\mathcal{I} - lim\, u_n = u$ and $G\mathcal{I} - lim\, v_n = v$, we have

$$\left\{(j,k) \in \mathbb{N}^2 : G(u, u_j, u_k) \geq \frac{\varepsilon}{4}\right\} \in \mathcal{I}_2,$$

$$\left\{(j,k) \in \mathbb{N}^2 : G(v, v_j, v_k) \geq \frac{\varepsilon}{4}\right\} \in \mathcal{I}_2.$$

After that

$$\begin{aligned}&\{(j,k) \in \mathbb{N}^2 : G(u+v, u_j+v_j, u_k+v_k) \geq \varepsilon\} \\ &\qquad \subseteq \left[\left\{(j,k) \in \mathbb{N}^2 : G(u, u_j, u_k) \geq \frac{\varepsilon}{4}\right\} \cup \left\{(j,k) \in \mathbb{N}^2 : G(v, v_j, v_k) \geq \frac{\varepsilon}{4}\right\}\right]\end{aligned}$$

can be easily verified. Thus, by properties of ideal,

$$\{(j,k) \in \mathbb{N}^2 : G(u+v, u_j+v_j, u_k+v_k) \geq \varepsilon\} \in \mathcal{I}_2,$$

i.e., $G\mathcal{I} - lim\, (u_n + v_n) = u + v$.

(ii) Similarly, it can be proved.

Definition 2.6: A sequence (u_n) of elements of X is said to be $G\mathcal{I}^*$-convergent to $u \in X$ if and only if there exists a set $M^* \in \mathcal{F}(\mathcal{I}_2)$,

$$M^* = \left\{ \left(m_{k_1}, m_{k_2} \right) : m_{k_i} \in M, i = 1,2 \right\} \subset \mathbb{N}^2$$

where $M \in \mathcal{F}(\mathcal{I})$, $M = \{ m_1 < m_2 < \cdots < m_k < \cdots \} \subset \mathbb{N}$ such that

$$\lim_{k_1, k_2 \to \infty} G\left(u, u_{m_{k_1}}, u_{m_{k_2}}\right) = 0.$$

The element u is called the $G\mathcal{I}^*$-limit of the sequence (u_n) and the notation $G\mathcal{I}^* - lim u_n = u$ is used.

Theorem 2.7: Let $\mathcal{I}_2$ be an admissible ideal. If $G\mathcal{I}^* - lim u_n = u$, then $G\mathcal{I} - lim u_n = u$.

Proof: By assumption there is a set $N \in \mathcal{I}_2$ such that

$$M^* = \mathbb{N}^2 \setminus \mathrm{N} = \left\{ \left(m_{k_1}, m_{k_2} \right) : m_{k_i} \in M, i = 1,2 \right\}$$

where $M \in \mathcal{F}(\mathcal{I})$, $M = \{ m_1 < m_2 < \cdots < m_k < \cdots \} \subset \mathbb{N}$. We have

$$\lim_{k_1, k_2 \to \infty} G\left(u, u_{m_{k_1}}, u_{m_{k_2}}\right) = 0. \tag{3}$$

Let $\varepsilon > 0$. By (3) there exists $m_{k_0} \in \mathbb{N}$ such that

$$G\left(u, u_{m_{k_1}}, u_{m_{k_2}}\right) < \varepsilon$$

for each $m_{k_1}, m_{k_2} \geq m_{k_0}$. Then

$$\begin{aligned} A &= \left\{ (j,k) \in \mathbb{N}^2 : G\left(u, u_j, u_k\right) \geq \varepsilon \right\} \\ &\subset \left[N \cup \left\{ \left(m_{k_1}, m_{k_2} \right) : m_{k_i} \in \left\{ m_1 < m_2 < \cdots < m_{k_0} \right\} \right\} \right]. \end{aligned} \tag{4}$$

Since $\mathcal{I}_2$ is an admissible and $N \in \mathcal{I}_2$, the set on the right-hand side of (4) belongs to $\mathcal{I}_2$. Hence $A \in \mathcal{I}_2$.

Now we define a necessary and sufficient property for the equality of $G\mathcal{I}$-convergence and $G\mathcal{I}^*$-convergence.

Definition 2.8: An admissible ideal $\mathcal{I}_2$ is said to satisfy the property (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, \dots\}$ belongs to $\mathcal{I}_2$ there exists countable family of sets $\{B_1, B_2, \dots\}$ such that $A_j \triangle B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$.

Remark 2.9: Note that also $B_j \in \mathcal{I}_2$ for $j \in \mathbb{N}$.

Theorem 2.10: Let (u_n) be a sequence in X and $\mathcal{I}_2$ be an admissible ideal with property (AP). If $G\mathcal{I} - lim u_n = u$, then $G\mathcal{I}^* - lim u_n = u$.

Proof: Assume that $\mathcal{I}_2$ satisfies property (AP). Let $G\mathcal{I} - lim u_n = u$ and $\varepsilon > 0$. Then

$$A_G(\varepsilon) = \{(j,k) \in \mathbb{N}^2 : G(u, u_j, u_k) \geq \varepsilon\} \in \mathcal{I}_2.$$

We can choose

$$A_1 = \{(j,k) \in \mathbb{N}^2 : G(u, u_j, u_k) \geq 1\}$$

and

$$A_n = \left\{(j,k) \in \mathbb{N}^2 : \frac{1}{n} \leq G(u, u_j, u_k) < \frac{1}{n-1}\right\}$$

for $n \geq 2, n \in \mathbb{N}$. Obviously $A_i \cap A_j \neq \emptyset$ for $i \neq j$. By property (AP) there exists a sequence of sets $\{B_n\}$ such that $A_j \triangle B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$. It is sufficient to prove that for $M^* = \mathbb{N}^2 \setminus \mathrm{B}$ we have

$$G - lim u_n = u. \tag{5}$$

Let $\delta > 0$. Choose $k \in \mathbb{N}$ such that $\frac{1}{k+1} < \delta$. Then

$$\{(j,k) \in \mathbb{N}^2 : G(u, u_j, u_k) \geq \delta\} \subset \bigcup_{j=1}^{k+1} A_j.$$

Since $A_j \triangle B_j, j = 1,2, \dots, k+1$ are finite sets, there exists $n_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{k+1} A_j\right) \cap \{(j,k) \in \mathbb{N}^2 : j,k > n_0\} = \left(\bigcup_{j=1}^{k+1} B_j\right) \cap \{(j,k) \in \mathbb{N}^2 : j,k > n_0\}. \tag{6}$$

If $j,k > n_0$ and $j,k \notin B$, then $j,k \notin \bigcup_{j=1}^{k+1} B_j$ and, by (6), $j,k \notin \bigcup_{j=1}^{k+1} A_j$ but then

$$G(u, u_j, u_k) < \frac{1}{k+1} < \delta.$$

Hence (5) holds.

Definition 2.11: Let $\mathcal{I}_2$ be an admissible ideal. Then a (u_n) sequence of elements of X is said to be $G\mathcal{I}$-Cauchy sequence in X if for every $\varepsilon > 0$ there exists $l \in \mathbb{N}$ such that

$$A_G(\varepsilon) = \{(j,k) \in \mathbb{N}^2 : G(u_j, u_k, u_l) \geq \varepsilon\} \in \mathcal{I}_2.$$

Definition 2.12: Let $\mathcal{I}_2$ be an admissible ideal. Then a (u_n) sequence of elements of X is said to be $G\mathcal{I}^*$-Cauchy sequence in X if for every $\varepsilon > 0$ there exists a set

$$M^* = \{(m_{k_1}, m_{k_2}) : m_{k_i} \in M, i = 1,2\} \subset \mathbb{N}^2$$

where $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that the subsequence $(u_M) = (u_{m_k})$ is an ordinary G-Cauchy sequence in X, i.e.,

$$\lim_{k_1,k_2,k_3 \to \infty} G\left(u_{m_{k_1}}, u_{m_{k_2}}, u_{m_{k_3}}\right) = 0.$$

Theorem 2.13: Let $\mathcal{I}_2$ be an arbitrary admissible ideal. Then $G\mathcal{I} - lim\, u_n = u$ implies that (u_n) is a $G\mathcal{I}$-Cauchy sequence.

Proof: Let $G\mathcal{I} - lim\, u_n = u$. Then for every $\varepsilon > 0$, we have

$$A_G(\varepsilon) = \left\{(j,k) \in \mathbb{N}^2 : G(u, u_j, u_k) \geq \frac{\varepsilon}{6}\right\} \in \mathcal{I}_2.$$

Since $\mathcal{I}_2$ is an admissible ideal, there exists an $l \in \mathbb{N}$ such that $l \notin A_G(\varepsilon)$. Let

$$B_G(\varepsilon) = \{(j,k) \in \mathbb{N}^2 : G(u_j, u_k, u_l) \geq \varepsilon\}.$$

By the properties of the G-metric space,

$$\begin{aligned} G(u_j, u_k, u_l) &\leq G(u_j, u, u) + G(u, u_k, u) + G(u, u, u_l) \\ &\leq 2[G(u, u_j, u_j) + G(u, u_k, u_k) + G(u, u_l, u_l)] \\ &\leq 2[G(u, u_j, u_k) + G(u, u_j, u_k) + G(u, u_j, u_l)] \\ &< 2\left(\frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6}\right) = \varepsilon \end{aligned}$$

Looking at it the other way, as $l \notin A_G(\varepsilon)$. we have $G(u_j, u_k, u_l) < \varepsilon$. This is a contradiction. Here we conclude that $B_G(\varepsilon) \subset A_G(\varepsilon) \in \mathcal{J}_2$ for every $\varepsilon > 0$. Thus $B_G(\varepsilon) \in \mathcal{J}_2$, i.e., (u_n) is a $G\mathcal{J}$-Cauchy sequence.

Theorem 2.14: Let $\mathcal{J}_2$ be an admissible ideal. If (u_n) is a $G\mathcal{J}^*$-Cauchy sequence then this sequence is $G\mathcal{J}$-Cauchy sequence.

Proof: Let (u_n) be a $G\mathcal{J}^*$-Cauchy sequence. Then there exists a set

$$M^* = \left\{(m_{k_1}, m_{k_2}) : m_{k_i} \in M, i = 1,2\right\} \subset \mathbb{N}^2$$

where $M \in \mathcal{F}(\mathcal{J})$, $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that

$$\mathrm{G}\left(u_{m_{k_1}}, u_{m_{k_2}}, u_{m_{k_3}}\right) < \varepsilon$$

for every $\varepsilon > 0$ and for all $m_{k_1}, m_{k_2}, m_{k_3} \geq m_{k_0}(\varepsilon)$. Let $N = N(\varepsilon) = m_{k_0+1}$. Then for every $\varepsilon > 0$, we get

$$\mathrm{G}\left(u_N, u_{m_{k_1}}, u_{m_{k_2}}\right) < \varepsilon, m_{k_1}, m_{k_2} \geq m_{k_0}$$

Now let $L = \mathbb{N}^2 \setminus M^*$. It is clear that $L \in \mathcal{J}_2$ and

$$A_G(\varepsilon) = \left\{(j,k) \in \mathbb{N}^2 : G(u_N, u_j, u_k) \geq \varepsilon\right\} \subset \left(L \cup \left\{(m_{k_1}, m_{k_2}) : m_{k_i} \in \left\{m_1 < m_2 < \cdots < m_{k_0}\right\}\right\}\right).$$

Then $\left(L \cup \left\{(m_{k_1}, m_{k_2}) : m_{k_i} \in \left\{m_1 < m_2 < \cdots < m_{k_0}\right\}\right\}\right) \in \mathcal{J}_2$. Thus, for every $\varepsilon > 0$ we can find an $N \in \mathbb{N}$ such that $A_G(\varepsilon) \in \mathcal{J}_2$. Hence (u_n) is $G\mathcal{J}$-Cauchy sequence.

Lemma 2.15: Let $\{S_i\}_{i=1}^{\infty}$ be a countable family of subsets of $\mathbb{N}^2$ such that $S_i \in \mathcal{F}(\mathcal{J}_2)$ for each i, where $\mathcal{F}(\mathcal{J}_2)$ is a filter associate with an admissible ideal $\mathcal{J}_2$ with property (AP). Then there exists a set $S \subset \mathbb{N}^2$ such that $S \in \mathcal{F}(\mathcal{J}_2)$ and the set $S \setminus S_i$ is finite for all i.

Proof: Let

$$\begin{aligned}
A_1 &= \mathbb{N}^2 \setminus S_1 \\
A_2 &= (\mathbb{N}^2 \setminus S_2) \setminus A_1 \\
&\vdots \\
A_m &= (\mathbb{N}^2 \setminus S_m) \setminus (A_1 \cup A_2 \cup \ldots \cup A_{m-1})
\end{aligned}$$

for $m = 2,3,\ldots$. It is easy to see that $A_i \in \mathcal{I}_2$ for each i and $A_i \cap A_j = \emptyset$, when $i \neq j$. Then by propery (AP) of $\mathcal{I}_2$ we conclude that there exists a countable family of sets $\{B_1, B_2\}$ such that $A_j \triangle B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$. Put $S = \mathbb{N}^2 \setminus B$. It is clear that $S \in \mathcal{F}(\mathcal{I}_2)$. Now prove that the set $S \setminus S_i$ is finite for all i. Assume that there exists a $j_0 \in \mathbb{N}$ such that $S \setminus S_{j_0}$ has infinitely many elements. Since each $A_j \triangle B_j$ is a finite set $(j = 1,2,\ldots,j_0)$, there exists a $n_0 \in \mathbb{N}$ such that

$$\bigcup_{j=1}^{j_0} B_j \cap \{(k,l) \in \mathbb{N}^2 : k,l > n_0\} = \bigcup_{j=1}^{j_0} A_j \cap \{(k,l) \in \mathbb{N}^2 : k,l > n_0\}.$$

If $k,l > n_0$ and $k,l \notin B$, then $k,l \notin \bigcup_{j=1}^{j_0} B_j$ and $k,l \notin \bigcup_{j=1}^{j_0} A_j$. Since $A_{j_0} = \left(\mathbb{N}^2 \setminus S_{j_0}\right) \setminus \bigcup_{j=1}^{j_0-1} A_j$ and $k,l \notin A_{j_0}$ we have $k,l \in S_{j_0}$ for $k,l > n_0$. Therefore, for all $k,l > n_0$ we get $k,l > n_0 \in S$ and $k,l > n_0 \in S_{j_0}$. This shows that the set $S \setminus S_{j_0}$ has a finite number of elements. This contradicts to our assumption that the set $S \setminus S_{j_0}$ is an infinite set. Hence the proof is complete.

Theorem 2.16: If $\mathcal{I}_2$ is an ideal that is admissible and has the property (AP), then the terms $G\mathcal{I}$-Cauchy sequence and $G\mathcal{I}^*$-Cauchy sequence are equivalent.

Proof: If a sequence (u_n) is $G\mathcal{I}^*$-Cauchy sequence, then it is $G\mathcal{I}$-Cauchy sequence by Theorem 2.14. where $\mathcal{I}_2$ need not have the property (AP). Now that it is assumed that (u_n) is a $G\mathcal{I}$-Cauchy sequence, it is sufficient to demonstrate that (u_n) is a $G\mathcal{I}^*$-Cauchy sequence.

Let (u_n) in X be a $G\mathcal{I}$-Cauchy sequence. Then, for every $\varepsilon > 0$ there exists $l \in \mathbb{N}$ such that

$$A_G(\varepsilon) = \left\{(j,k) \in \mathbb{N}^2 : G\left(u_j, u_k, u_l\right) \geq \varepsilon\right\} \in \mathcal{I}_2.$$

Let

$$S_i = \left\{(j,k) \in \mathbb{N}^2 : G\left(u_{m_{k_i}}, u_j, u_k\right) < \frac{1}{i}\right\},$$

$i = 1,2,\ldots$ where $m_{k_i} = l\left(\frac{1}{i}\right)$. It is clear that $S_i \in \mathcal{F}(\mathcal{I}_2)$ for $i = 1,2,\ldots$. Since $\mathcal{I}_2$ has the property (AP), then by Lemma 2.15. there exists a set $S \subset \mathbb{N}^2$ such that $S \in \mathcal{F}(\mathcal{I}_2)$ and $S \setminus S_i$ is finite for all i. Now we show that

$$\lim_{k_1,k_2,k_3 \to \infty} G\left(u_{m_{k_1}}, u_{m_{k_2}}, u_{m_{k_3}}\right) = 0.$$

Let $\varepsilon > 0$ and $j_0 \in \mathbb{N}$ such that $j_0 > \frac{3}{\varepsilon}$. If $k_1, k_2, k_3 \in S$ then $S \setminus S_{j_0}$ is a finite set, so there exists $p = p(j_0)$ such that $k_1, k_2, k_3 \in S_{j_0}$ for all $k_1, k_2, k_3 \geq p(j_0)$. Thus it follows that

$$\begin{aligned} G\left(u_{m_{k_1}}, u_{m_{k_2}}, u_{m_{k_3}}\right) &\leq G\left(u_{m_{k_1}}, u_j, u_j\right) + G\left(u_{m_{k_2}}, u_j, u_j\right) + G\left(u_{m_{k_3}}, u_j, u_j\right) \\ &\leq G\left(u_{m_{k_1}}, u_j, u_k\right) + G\left(u_{m_{k_2}}, u_j, u_k\right) + G\left(u_{m_{k_3}}, u_j, u_k\right) \\ &\leq \frac{1}{j_0} + \frac{1}{j_0} + \frac{1}{j_0} < \varepsilon \end{aligned}$$

This demonstrates that the sequence (u_n) in X is a $G\mathcal{I}^*$-Cauchy sequence.

III. Conclusion and Recommendations

Gahler was the first to conduct research on generalizing metric spaces (Gahler, 1963). These investigations led to the elimination of several axioms' shortcomings and Mustafa and Sims' proper extension of the idea of metric as it was previously understood (Mustafa & Sims, 2006). Kolancı and Gürdal (Kolancı & Gürdal, 2023) provided a more comprehensive description known as the G-statistical convergence and $G\mathcal{I}$-convergence with the aid of Mustafa and Sims' work. Three contributions are made to the area of summability theory by this study: (i) the notion of $G\mathcal{I}$ -Cauchy and $G\mathcal{I}^*$-Cauchy sequences in G-metric spaces; (ii) the $G\mathcal{I}^*$-convergence in G-metric spaces; and (iii) a form of ideal convergence in G-metric spaces. These findings can be applied to the research of double sequence convergence issues.

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References

Abazari, R. (2022). Statistical convergence in g-metric spaces. Filomat, 36(5), 1461-1468.

Choi, H., Kim, S., & Yang, S. (2018). Structure for g-metric spaces and related fixed point theorems. Arxive: 1804.03651v1.

Demirci, I., Kişi, Ö., & Gürdal, M. (2023). Rough I_2-statistical convergence in cone metric spaces in certain details. Bulletin of Mathematical Analysis and Applications, 15(1), 7-23.

Dhage, B. (1992). Generalized metric space and mapping with fixed point. Bulletin of the Calcutta Mathematical Society, 84, 329-336.

Fast, H. (1951). Sur la convergence statistique. Colloquium Mathematicae, 2, 241-244.

Fridy, J. (1985). On statistical convergence. Analysis, 5(4), 301-314.

Gahler, S. (1963). 2-metriche raume und ihre topologische strüktüre. Mathematische Nachrichten, 26, 115-148.

Gahler, S. (1966). Zur geometric 2-metriche raume. Revue Roumaine de Mathematiques Pures et Appliquees, 40, 664-669.

Gürdal, M., Kişi, Ö., & Kolancı, S. (2023). New convergence definitions for double sequences in g-metric spaces. Journal of Classical Analysis, 21(2), 173-185.

Gürdal, M., Kolancı, S., & Kişi, Ö. (2023). On generalized statistical convergence in g-metric spaces. Ilirias Journal of Mathematics.

Gürdal, V., & Kişi, Ö. (2022). Some properties of I3-λ-statistical cluster points. Dera Natung Government College Research Journal, 7(1), 10-19.

Kolancı, S., & Gürdal, M. (2023). G-metrik uzaylarda bazı yakınsaklık kavramları. 2nd International Conference Contemporary Academiz Research (s. 259-263). Konya: AllSciences Proceedings.

Kolancı, S., Gürdal, M., & Kişi, Ö. (2023). g-metric spaces and asymptotically lacunary statistical equivalent sequences. Honam Mathematical Journal, 45(3), 503-512.

Kostyrko, P., Salat, T., & Wilczynski, W. (2000). I-convergence. Real Analysis Exchange, 26(2), 669-686.

Mustafa, Z., & Sims, B. (2003). Some remarks concerning D-metric spaces. Proceedings of the Internatinal Conferences on Fixed Point Theorey and Applications, (s. 189-198). Valencia (Spain).

Mustafa, Z., & Sims, B. (2006). A new approach to generalized metric spaces. Journal of Nonlinear and Convex Analysis, 7(2), 289-297.

Nabiev, A., Pehlivan, S., & Gürdal, M. (2007). On I-Cauchy sequences. Taiwanese Journal of Mathematics, 11(2), 569-576.

Nabiev, A., Savaş, E., & Gürdal, M. (2019). Statistically localized sequences in metric spaces. Journal of Applied Analysis and Computation, 9(2), 739-746.

Savaş, E., & Gürdal, M. (2015). I-statistical convergence in probabilistic normed spaces. Scientific Bulletin-Series A Applied Mathematics and Physics, 77(4), 195-204.

Steinhaus, H. (1951). Sur la convergence ordinaire et la convergence asymptotiqu. Colloquium Mathematicae, 2, 73-74.

Şahiner, A., Gürdal, M., & Yiğit, T. (2011). Ideal convergence characterization of the completion of linear n-normed spaces. Computers & Mathematics with Applications, 61(3), 683-689.

Yamancı, U., & Gürdal, M. (2013). On lacunary ideal convergence in random-normed space. Journal of Mathematics, 2013, Article ID 868457, 8 pages.