Dera Natung Government College Research Journal
Volume 9 Issue 1, pp. 13-27, January-December 2024

ISSN (Print): 2456-8228
ISSN (Online): 2583-5483

Research Article

Paranormed spaces of absolute tribonacci summable series and some matrix transformations

Fadime Gökçe*

Faculty of Science, University of Pamukkale, Denizli, Turkey.

Cite as: Gökçe, F. (2024). Paranormed spaces of absolute tribonacci summable series and some matrix transformations, Dera Natung Government College Research Journal, 9, 13-27.
https://doi.org/10.56405/dngcrj.2024.09.01.02

Received on: 18.01.2024,
Revised on: 13.03.2024,
Accepted on: 09.07.2024,
Available online: 30.12.2024.

*Corresponding Author: **Fadime Gökçe** (fgokce@pau.edu.tr)

Abstract: In a more recent paper, the absolute series space $|T_\varphi|_q$ which is defined as the domain of the matrix corresponding to the absolute Tribonacci summability in the well-known space l_q has taken place in the literature **(Gökçe, 2025)**. The present study is mainly aimed to establish the absolute series space $|T_\varphi|(\delta)$ which includes $|T_\varphi|_q$, as the set of all series summable by the absolute Tribonacci method in $l(\delta)$, and to investigate its some topological and algebraic properties. Moreover, certain characterizations of matrix operators on this space is obtained.

Keywords: Absolute Summability, Matrix transformations, Maddox's space, Tribonacci numbers.

MSC 2020: 40C05, 40F05, 46A45.

1. Introduction

The theory of summability, which has been used in many fields of science from the past to the present, has expanded with the sequence spaces generated by summability methods and matrix transformations on these spaces on the one hand, and on the other hand, it has continued to progress with the study of new series spaces defined by absolute summability methods and some matrix transformations related to them (see **(Dağlı and Yaying, 2023)**, **(Gökçe, 2021; 2022)**, **(Gökçe and Sarıgöl, 2018; 2018a; 2019; 2019a; 2020; 2020a; 2020b)**, **(Sarıgöl, 2010)**, **(Yaying and Kara, 2021)**, **(Yaying and Hazarika, 2020)**. In this paper, the absolute series space $|T_\varphi|(\delta)$ is established as the set of all series summable by the absolute Tribonacci method, and its some algebraic and topological structure such as FK-space, duals and Schauder base are given. Furthermore, certain characterizations of matrix operators on the space are obtained. Firstly, let us remind some basic concepts.

By ω, l_∞, c and l_q $(q \geq 1)$, we stand for the set of all sequences of complex numbers, the sequence spaces of all bounded, convergent sequences and also the space of all q-absolutely convergent series. Besides, throughout the paper, $\mathbb{N} = \{0,1,2,3,...\}$. Let $\Lambda = (\lambda_{ji})$ be an arbitrary infinite matrix of complex components and U, V be two subspaces of ω. If the series

$$\Lambda_j(u) = \sum_{i=0}^{\infty} \lambda_{ji} u_i$$

converges for all $j \epsilon \mathbb{N}$, then, it is said that the Λ -transform of the sequence $u = (u_i)$ is identified by $\Lambda(u) = (\Lambda_j(u))$. Also, it is said that Λ describes a matrix transformation from the space U into the space V, and the class of all infinite matrices $\Lambda : U \to V$ is represented by (U, V).

If $r_{ji} = 0$ for $i > j$ and otherwise $r_{ji} \neq 0$ for all j, i, then it is said that R is a triangle.

The notion of the multiplier space of U and V is identified by

$$S(U,V) = \{ z = (z_j) \in \omega : \forall\, u \in U, uz = (u_j z_j) \in V \},$$

and according to the notation of multiplier space, the α, β and γ duals of U are defined as

$$U^\alpha = S(U,l) = \left\{ z = (z_j) \in \omega : \forall\, u = (u_j) \in U, \sum_{j=0}^{\infty} |u_j z_j| < \infty \right\},$$

$$U^\beta = S(U,c_s) = \left\{ z = (z_j) \in \omega : \forall\, u = (u_j) \in U, \left(\sum_{j=0}^{n} u_j z_j \right) \in c \right\},$$

$$U^\gamma = S(U,b_s) = \left\{ z = (z_j) \in \omega : \forall\, u = (u_j) \in U, \left(\sum_{j=0}^{n} u_j z_j \right) \in l_\infty \right\}.$$

Here, c_s and b_s represent the set of all convergent and bounded series, respectively.

The set

$$U_{\boldsymbol{\Lambda}} = \{ u = (u_j) \in \omega : \Lambda(u) \in \mathrm{U} \}$$

defines domain of an infinite matrix Λ in U. This is another important concept. It is clear that the set also determines a sequence space.

A linear topological space (also called a topological vector space) is both of a topological space and a vector space such that the properties of scalar multiplication and vector addition are continuous. Assume that U is a topological vector space over $\mathbb{R}$. For all $a \in \mathbb{R}$ and $\tilde{u}, u \in U$, if

- $f: U \to \mathbb{R}$ such that $f(0) = 0, f(u) = f(-u)$
- $f(u + \tilde{u}) \leq f(u) + f(\tilde{u})$

- $|a_n - a| \to 0$, $f(u_n - u) \to 0$ imply $f(a_n u_n - au) \to 0$ as $n \to \infty$, that is the scalar multiplication is continuous

then, it is said that U is a paranormed space.

Let $U \subset \omega$. If it is a Frechet space with continuous coordinates $R_j: U \to \mathbb{C}$, where $R_j(u) = u_j$ for all $u \in U, j \in \mathbb{N}$, then it is said that U is an FK-space. Also, an FK-space whose metric also determines a norm is said to be a BK-space. In several areas of summability theory, these concepts play an important role, for instance, all matrix transformations from an FK-space to another FK-space are continuous **(Wilansky, 1984)**. If there exists unique sequence of coefficient (u_i) such that

$$\lim_{j\to\infty} \sum_{i=0}^{j} u_i b_i = u,$$

for all $u \in U$, then (b_i) is called a Schauder base of an FK-space U. For an example to these concepts, it can be said that the sequence $\left(e^{(i)}\right)$ whose terms given by

$$e_j^{(i)} = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases}$$

for $i \geq 0$, is the Schauder base of the Maddox space $l(\delta)$. The Maddox space $l(\delta)$ can be expressed clearly as follows:

$$l(\delta) = \left\{ u = (u_j): \sum_{j=0}^{\infty} |u_j|^{\delta_j} < \infty \right\}.$$

With the natural paranorm of the space

$$f(u) = \left(\sum_{j=0}^{\infty} |u_j|^{\delta_j} \right)^{\frac{1}{P}},$$

the Maddox space $l(\delta)$ is an FK-space where $P = max\,\{1, \sup_j \delta_j\}$. In addition to this, in the case of $\delta_j \geq 1$ for each j, the space becomes a BK- space with the following norm (see **(Maddox, 1967;1968;1969)**)

$$\|u\| = inf \left\{ \varrho > 0: \sum_{j=0}^{\infty} |u_j/\varrho|^{\delta_j} \leq 1 \right\}.$$

Unless otherwise stated, throughout the whole study, we assume that $\Lambda = (\lambda_{ji})$ is an infinite matrix of complex components for all $j, i \in \mathbb{N}$, (ψ_j) and (φ_j) are any sequences of positive numbers, $\eta = (\eta_j)$ and, $\delta = (\delta_j)$ are bounded sequences of positive real numbers with $0 < inf\eta_j \leq h < \infty, 0 < inf\delta_j \leq m < \infty$ and $1/\delta_j + 1/\delta_j^* = 1$ for $\delta_j > 0$, $1/\delta_j^* = 0$ for $\delta_j = 1$.

On the other hand, Tribonacci numbers, which constitute the other branch of the study are very interesting. Tribonacci numbers determine a sequence of integers defined by the third order recurrence relation with initial conditions

$$t_0 = 1, t_1 = 1, t_2 = 2,$$
$$t_i = t_{i-1} + t_{i-2} + t_{i-3},$$
$$t_{-i} = 0, i > 0.$$

So, some of the first Tribonacci numbers as follows:

$$1, 1, 2, 4, 7, 13, 24, 44 \cdots.$$

Furthermore, Tribonacci numbers have the following useful properties:

$$\sum_{i=0}^{j} t_i = \frac{t_{j+2} + t_j - 1}{2},$$

$$\sum_{i=0}^{j} t_{2i} = \frac{t_{2j+1} + t_{2j} - 1}{2},$$

$$\lim_{j\to\infty} \frac{t_j}{t_{j+1}} = 0.54368901 \ldots$$

In addition to these properties, Tribonacci matrix $T = (t_{ji})$ has recently been defined by Yaying and Hazarika **(2020)** as follows:

$$t_{ji} = \begin{cases} \dfrac{2t_i}{t_{j+2} + t_j - 1}, & 0 \leq i \leq j \\ 0, & i > j \end{cases}$$

where t_i is the ith Tribonacci number for all $i \epsilon \mathbb{N}$.

Let take the series $\sum u_i$ with the sequence of its jth partial sum $s = (s_j)$, and let $\varphi = (\varphi_j)$ be arbitrary sequence of positive real numbers, $\delta = (\delta_j)$ be a bounded sequence of positive real numbers. If

$$\sum_{j=1}^{\infty} \varphi_j^{\delta_{j-1}} |\Lambda_j (s) - \Lambda_{j-1} (s)|^{\delta_j} < \infty,$$

the series $\sum u_i$ is said to be summable $|\Lambda, \varphi_j|(\delta)$ **(Gökçe & Sarıgöl, 2018a)**. Obviously, the summability method $|\Lambda, \varphi_j|(\delta)$ is a very comprehensive method such that it includes a number of well-known absolute summability methods for special choices of the matrix Λ and the sequences φ, δ. To give a few examples, we can present: if we chose just a triangle matrix T instead of Λ, the summability method $|T, \varphi_j|(\delta)$ is immediately obtained with $\delta_j = k$ for all j **(Gökçe, 2022)**, again if we select the Euler, Nörlund, Cesàro, the weighted mean matrices instead of U, the summability method $|\Lambda, \varphi_j|(\delta)$ is reduced to the summabilities $|E^r, \varphi_j|(\delta)$, $|N, p_j, \varphi_j|(\delta)$, $|C, \alpha, \beta|(\delta)$ (with $\varphi_j = j$ for all j), $|\bar{N}, p_j, \varphi_j|(\delta)$ **(Gökçe & Sarıgöl, 2018; 2018a; 2019; 2019a)**, respectively, (see also **(Gökçe, 2021; 2022)**, **(Gökçe and Sarıgöl, 2020; 2020a; 2020b)**).

Finally, we present some lemmas that will be used in the proofs and then move on to the main parts:

Lemma 1.1. **(Grosse-Erdmann, 1993)** *Let $\delta = (\delta_i)$ and $\eta = (\eta_i)$ be any bounded sequences of strictly positive numbers.*

(i) *If $\delta_i > 1$ for each i, then, $\Lambda \in (l(\delta), l)$ iff there exists an integer $c > 1$ such that*

$$sup\left\{\sum_{i=0}^{\infty}\left|\sum_{j\epsilon G}\lambda_{ji}\ c^{-1}\right|^{\delta_i^*} : G \subset \mathbb{N}\ finite\right\} < \infty. \tag{1}$$

(ii) *If $\eta_i \geq 1$ and $\delta_i \leq 1$ for each $i \in \mathbb{N}$ then, $\Lambda \in (l(\delta), l(\eta))$ iff there exists some c such that*

$$\sup_i \sum_{j=0}^{\infty}\left|\lambda_{ji}\ c^{-1/\delta_i}\right|^{\eta_j} < \infty.$$

(iii) *If $\delta_i \leq 1$ for all i, then,*

$$\Lambda \in (l(\delta), c) \Leftrightarrow \begin{cases} (a)\ \lim_{j\to\infty}\lambda_{ji}\ exists\ for\ each\ i \\ (b)\ \sup_{j,i}|\lambda_{ji}|^{\delta_i} < \infty \end{cases}$$

$$\Lambda \in (l(\delta), c_0) \Leftrightarrow \begin{cases} (c)\ \lim_{j\to\infty}\lambda_{ji} = 0\ \ for\ each\ i, \\ (b)\ holds \end{cases}$$

$$\Lambda \in (l(\delta), l_\infty) \Leftrightarrow (b)\ holds.$$

(iv) *If $\delta_i > 1$ for all i, then,*

$$\Lambda \in (l(\delta), c) \Leftrightarrow (a')\ \lim_{j\to\infty}\lambda_{ji}\ exists\ for\ each\ i,\quad (b')\ there\ exists\ a\ number\ c > 1\ such\ that$$

$$\sup_j \sum_{i=0}^{\infty}\left|\lambda_{ji}c^{-1}\right|^{\delta_i^*} < \infty,$$

$$\Lambda \in (l(\delta), c_0) \Leftrightarrow (c') \lim_{j\to\infty} \lambda_{ji} = 0 \text{ for each } i, \qquad (b') \text{ holds}$$

$$\Lambda \in (l(\delta), c_0) \Leftrightarrow (b') \text{ holds.}$$

It should be noted that condition (1) presents a number of difficulties in terms of its application. However, Lemma 1.2, which gives a condition equivalent to (1) and is more practical in most cases, will be preferred in the proofs of our theorems.

Lemma 1.2. **(Sarıgöl, 2013)** Let $\Lambda = (\lambda_{ji})$ be an infinite matrix with complex components, $\delta = (\delta_i)$ be a bounded sequence of positive numbers. If $W_\delta[\Lambda] < \infty$ or $L_\delta[\Lambda] < \infty$, then

$$(2m)^{-2} W_\delta[\Lambda] \le L_\delta[\Lambda] \le W_\delta[\Lambda],$$

where $m = \max\{1, 2^{M-1}\}, M = \sup_i \delta_i$.

$$W_\delta[\Lambda] = \sum_{i=0}^{\infty}\left(\sum_{j=0}|\lambda_{ji}|\right)^{\delta_i}$$

and

$$L_\delta[\Lambda] = \sup\left\{\sum_{i=0}^{\infty}\left|\sum_{j\in G}\lambda_{ji}\right|^{\delta_i} : G \subset \mathbb{N} \text{ finite}\right\}.$$

Lemma 1.3. **(Malkowsky & Rakocevic, 2000)** *Let R be a triangle. Then, for $U, V \subset \omega, \Lambda \in (U, V_R)$ iff $B = R\Lambda \in (U, V)$.*

Lemma 1.4. **(Malkowsky & Rakocevic, 2007)** *Let U be an FK-space with AK property, R be a triangle with its inverse S and $V \subset \omega$. Then, we have $\Lambda \in (U_R, V)$ if and only if $\tilde{\Lambda} \in (U, V)$ and $V^{(k)} \in (U, c)$ for all k, where*

$$\tilde{\lambda}_{kp} = \sum_{i=p}^{\infty} \lambda_{ki} s_{ip}, k, p = 0,1, \dots$$

$$v_{mp}^{(k)} = \begin{cases} \sum_{i=v}^{m} \lambda_{ki} s_{ip}, & 0 \le p \le m \\ 0, & p > m. \end{cases}$$

2. Main Results

In this part of the article, we will first establish the absolute Tribonacci summability method combining the concepts of absolute summability and Tribonacci matrix. To obtain this method, let us take the series $\sum u_i$ and its partial sums s_j. Then we get

$$\Lambda_m(s) = \sum_{j=0}^{m} \boldsymbol{t}_{mj} s_j = \sum_{i=0}^{m} u_i \sum_{j=i}^{m} \boldsymbol{t}_{mj} = \sum_{i=0}^{m} u_i \sum_{j=i}^{m} \frac{2t_j}{t_{m+2} + t_m - 1}$$

and so,

$$\Delta\Lambda_m(s) = \sum_{i=0}^{m} u_i \sum_{j=i}^{m} \frac{2t_j}{t_{m+2} + t_m - 1} - \sum_{i=0}^{m-1} u_i \sum_{j=i}^{m-1} \frac{2t_j}{t_{m+1} + t_{m-1} - 1}$$

$$= \frac{2t_m}{t_{m+2} + t_m - 1} u_m + \sum_{i=0}^{m-1} u_j \left(\frac{2t_m}{t_{m+2} + t_m - 1} + \Delta\sigma_m \sum_{j=i}^{m-1} 2t_j \right)$$

$$= \sum_{i=0}^{m} u_i \, \Phi_{mi}$$

where

$$\Phi_{mi} = \begin{cases} \dfrac{2t_m}{t_{m+2} + t_m - 1}, & i = m \\ \dfrac{2t_m}{t_{m+2} + t_m - 1} + \Delta\sigma_m \displaystyle\sum_{j=i}^{m-1} 2t_j\,, & 0 \le i \le m-1 \\ 0, & i > m, \end{cases}$$

$$\Delta\sigma_m = \sigma_m - \sigma_{m-1}, \sigma_m = \frac{1}{t_{m+2} + t_m - 1}.$$

With all of these informations, we can express the absolute Tribonacci series space as the set of all series summable by this method as follows:

$$|T_\varphi|(\delta) = \left\{ u \in \omega : \sum_{j=0}^{\infty} \varphi_j^{\delta_j - 1} \left| \sum_{i=0}^{j} u_i \, \Phi_{ji} \right|^{\delta_j} < \infty \right\}.$$

Furthermore, we can write

$$(E^{(\delta)} \circ \tilde{T})_j(u) = \varphi_j^{1/\delta_j^*} (\tilde{T}_j(u) - \tilde{T}_{j-1}(u))$$

where

$$\tilde{t}_{ji}=\begin{cases}\sigma_j\sum_{v=i}^{j}2t_v, & 0\le i\le j\\ 0, & i>j,\end{cases} \tag{2}$$

$$e_{ji}^{(\delta)}=\begin{cases}-\varphi_j^{1/\delta_j^*}, & i=j-1\\ \varphi_j^{1/\delta_j^*}, & i=j\\ 0, & i\neq j-1,j.\end{cases} \tag{3}$$

So, according to the notation of domain, the space redefines as $|T_\varphi|(\delta)=\big(l(\delta)\big)_{E^{(\delta)}\circ\tilde{T}}$.

Also, it is known that there exists a unique inverse matrix which also is a triangle for every triangle matrix. So, it can be easily seen that the terms of inverse of the matrices $\tilde{T}$and $E^{(\delta)}$ are as follows:

$$\tilde{t}_{ji}^{-1}=\begin{cases}\frac{1}{2\sigma_j t_j}, & i=j\\ -\frac{1}{2\sigma_{j-1}t_j}-\frac{1}{2\sigma_{j-1}t_{j-1}}, & i=j-1\\ \frac{1}{2\sigma_{j-2}t_{j-1}}, & i=j-2\\ 0, & otherwise,\end{cases}$$

$$\left(e_{ji}^{(\delta)}\right)^{-1}=\begin{cases}\varphi_i^{-1/\delta_i^*}, & 0\le i\le j\\ 0, & i>j.\end{cases}$$

From this point on, we can start to state and prove the following theorems which give some algebraic and topological properties of the mentioned space.

Theorem 2.1.

(a) *The set $|T_\varphi|(\delta)$ is a linear space with the scalar multiplication and coordinate-wise addition. Also, the space $|T_\varphi|(\delta)$ becomes an FK-space under the paranorm*

$$\|u\|_{|T_\varphi|(\delta)}=\|E^{(\delta)}\circ\tilde{T}(u)\|_{l(\delta)}=\left(\sum_{j=0}^{\infty}\left|\left(E^{(\delta)}\circ\tilde{T}(u)\right)_j\right|^{\delta_j}\right)^{\frac{1}{P}}$$

where $P=max\{1,\sup_j\delta_j\}$.

(b) *If $\delta_j = q$ for all $j\epsilon\mathbb{N}$, the space $|T_\varphi|(\delta)$ is a BK-space with the following norm*

$$\|u\|_{|T_\varphi|(\delta)} = \left\| E^{(q)} \circ \tilde{T}(u)\right\|_q.$$

Proof. The first part of the proof is a standard verification, so it is omitted. Since the matrices $E^{(\delta)}, \tilde{T}$ are triangles, it is clear that the composite function $E^{(\delta)} \circ \tilde{T}$ is also triangle. Furthermore, since $l(\delta)$ is an FK-space, then it can be written from Wilansky's Theorem 4.3.2 **(1984)** that $|T_\varphi|(\delta) = \big(l(\delta)\big)_{E^{(\delta)}\circ\tilde{T}}$ is also an FK-space. This concludes the proof.

A similar method can be used to prove the remaining part of the theorem.

Theorem 2.2. *The absolute series space $|T_\varphi|(\delta)$ has a Schauder basis $b^{(j)}$ whose terms are given by*

$$b_m^{(j)} = \begin{cases} \varphi_j^{-1/\delta_j^*}\left(\dfrac{1}{2\sigma_m t_m} - \dfrac{1}{2\sigma_{m-1} t_m} - \dfrac{1}{2\sigma_{m-1} t_{m-1}} - \dfrac{1}{2\sigma_{m-2} t_{m-1}}\right), & j \le m-2 \\ \varphi_{m-1}^{-1/\delta_{m-1}^*}\left(\dfrac{1}{2\sigma_m t_m} - \dfrac{1}{2\sigma_{m-1} t_m} - \dfrac{1}{2\sigma_{m-1} t_{m-1}}\right), & j = m-1 \\ \varphi_m^{-1/\delta_m^*}\dfrac{1}{2\sigma_m t_m}, & j = m \\ 0, & j > m. \end{cases}$$

Proof. It is noted that the sequence $(e^{(j)})$ is the Schauder base of the Maddox's space $l(\delta)$. So, it follows from Theorem 2.3 in **(Malkowsky & Rakocevic, 2007)**, $b^{(j)} = (\tilde{T}_m^{-1}((E^{(\delta)})^{-1}(e^{(j)}))$ determines the Schauder base of $|T_\varphi|(\delta)$.

Theorems 2.1 and 2.2 give us the result that the absolute series space $|T_\varphi|(\delta)$ is a separable space because the space is a linear metric space with a Schauder base.

Theorem 2.3. *The space $|T_\varphi|(\delta)$ is linearly isomorphic to the Maddox's space $l(\delta)$ i.e., $|T_\varphi|(\delta) \cong l(\delta)$.*

Proof. In order to prove the theorem, it should be shown that there is a linear bijection between $|T_\varphi|(\delta)$ and $\mathrm{l}(\delta)$. Let consider the transformations $\tilde{T}: |T_\varphi|(\delta) \to \big(l(\delta)\big)_{E^{(\delta)}}, E^{(\delta)}: \big(l(\delta)\big)_{E^{(\delta)}} \to l(\delta)$ and the matrices corresponding to them defined by (2) and (3). It is clear that the composite function $E^{(\delta)} \circ \tilde{T}$ is also triangle and so the composite function $E^{(\delta)} \circ \tilde{T}$ is a linear bijective operator. Moreover,

$$\|u\|_{|T_\varphi|(\delta)} = \left\| E^{(\delta)} \circ \tilde{T}(u)\right\|_{\mathrm{l}(\delta)}$$

that is, the paranorm is preserved. So, the proof is completed.

At this point, we describe the following sets and notation:

$$D_1=\left\{\varepsilon=(\varepsilon_j)\in\omega:\exists c>1\ ,\sum_{i=0}^{\infty}\frac{c^{-1/\delta_i^*}}{\varphi_i}\left(\sum_{j=i+2}^{\infty}\left|\varepsilon_j\left(\frac{1}{2t_j}\Delta\left(\frac{1}{\sigma_j}\right)-\frac{1}{2t_{j-1}}\Delta\left(\frac{1}{\sigma_{j-1}}\right)\right)\right|\right.\right.$$

$$\left.\left.+\left|\ \varepsilon_{i+1}\left(\frac{1}{2\sigma_{i+1}t_{i+1}}-\frac{1}{2\sigma_i t_{i+1}}-\frac{1}{2\sigma_i t_i}\right)\right|+\left|\frac{\varepsilon_i}{2\sigma_i t_i}\right|\right)^{\delta_i^*}<\infty\right\}$$

$$D_2=\left\{\varepsilon=(\varepsilon_j)\in\omega:\ \sup_i\left\{\varphi_i^{-1/\delta_i^*}\left(\sum_{j=i+2}^{\infty}\left|\varepsilon_j\left(\frac{1}{2t_j}\Delta\left(\frac{1}{\sigma_j}\right)-\frac{1}{2t_{j-1}}\Delta\left(\frac{1}{\sigma_{j-1}}\right)\right)\right|\right.\right.\right.$$

$$\left.\left.\left.+\left|\ \varepsilon_{i+1}\left(\frac{1}{2\sigma_{i+1}t_{i+1}}-\frac{1}{2\sigma_i t_{i+1}}-\frac{1}{2\sigma_i t_i}\right)\right|+\left|\frac{\varepsilon_i}{2\sigma_i t_i}\right|\right)\right\}<\infty\right\},$$

$$D_3=\left\{\varepsilon=(\varepsilon_j)\in\omega:\exists c>1:\ \sup_m\left(\frac{c^{-1/\delta_{m-1}^*}}{\varphi_{m-1}}|\xi_{m-1}|^{\delta_{m-1}^*}+\frac{c^{-1/\delta_m^*}}{\varphi_m}\left|\frac{\varepsilon_m}{2\sigma_m t_m}\right|^{\delta_m^*}\right.\right.$$

$$\left.\left.+\sum_{i=0}^{m-2}\frac{c^{-1/\delta_i^*}}{\varphi_i}\left|\xi_m+\sum_{j=i+2}^{m}\varepsilon_j\left(\frac{1}{2t_j}\Delta\left(\frac{1}{\sigma_j}\right)-\frac{1}{2t_{j-1}}\Delta\left(\frac{1}{\sigma_{j-1}}\right)\right)\right|^{\delta_i^*}\right)<\infty\right\},$$

$$D_4=\left\{\varepsilon=(\varepsilon_j)\in\omega\ :\ \sup_{m,i}\left(\left|\varphi_i^{-1/\delta_i^*}\left(\xi_i+\sum_{j=i+2}^{m}\left(\frac{1}{2t_j}\Delta\left(\frac{1}{\sigma_j}\right)-\frac{1}{2t_{j-1}}\Delta\left(\frac{1}{\sigma_{j-1}}\right)\right)\right)\right|^{\delta_i}\right.\right.$$

$$\left.\left.+\left|\varphi_{m-1}^{-\frac{1}{\delta_{m-1}^*}}\xi_{m-1}\right|^{\delta_{m-1}}+\left|\varphi_m^{-\frac{1}{\delta_m^*}}\frac{\varepsilon_m}{2\sigma_m t_m}\right|^{\delta_m}\right)<\infty\right\},$$

$$D_5=\left\{\varepsilon=(\varepsilon_j)\in\omega:\sum_{j=i+2}^{\infty}\varepsilon_j\left(\frac{1}{2t_j}\Delta\left(\frac{1}{\sigma_j}\right)-\frac{1}{2t_{j-1}}\Delta\left(\frac{1}{\sigma_{j-1}}\right)\right)\text{exist for all } i\right\},$$

$$\xi_i=\frac{\varepsilon_i}{2\sigma_i t_i}+\frac{\varepsilon_{i+1}}{2\sigma_{i+1}t_{i+1}}-\frac{\varepsilon_{i+1}}{2\sigma_i t_{i+1}}-\frac{\varepsilon_{i+1}}{2\sigma_i t_i}. \tag{4}$$

Similarly, when λ_{ji} is used instead of ε_i in (4), the notation $\xi_i^{(j)}$ will be used instead of ξ_i.

Theorem 2.4. *If $\delta_i>1$ for all i, then*

$$\{|T_\varphi|(\delta)\}^\alpha=D_1,\{|T_\varphi|(\delta)\}^\beta=D_3\cap D_5,\{|T_\varphi|(\delta)\}^\gamma=D_3$$

and if $\delta_i\le 1$ for all i, then,

$$\{|T_\varphi|(\delta)\}^\alpha=D_2,\{|T_\varphi|(\delta)\}^\beta=\ D_4\cap D_5,\{|T_\varphi|(\delta)\}^\gamma=D_4.$$

Proof. Considering the definition of the β dual, it can be written immediately that $\varepsilon\,\epsilon\{|T_\varphi|(\delta)\}^\beta$ if and only if $(\varepsilon_j u_j)\epsilon c_s$ for all $u\epsilon|T_\varphi|(\delta)$. It follows from the inverses of $\tilde{T}$ and $E^{(\delta)}$ that

$$\sum_{j=0}^{m}\varepsilon_j u_j=\sum_{j=0}^{m}\varepsilon_j\left(\frac{y_j}{2\sigma_j t_j}-\frac{y_{j-1}}{2\sigma_{j-1}t_j}-\frac{y_{j-1}}{2\sigma_{j-1}t_{j-1}}+\frac{y_{j-2}}{2\sigma_{j-2}t_{j-1}}\right)$$

$$= \sum_{i=0}^{m}\sum_{j=i}^{m} \varphi_i^{-1/\delta_i^*} \frac{\varepsilon_j z_i}{2\sigma_j t_j} - \sum_{i=0}^{m-1}\sum_{j=i+1}^{m} \varphi_i^{-1/\delta_i^*} \frac{\varepsilon_j z_i}{2\sigma_{j-1} t_j} - \sum_{i=0}^{m-1}\sum_{j=i+1}^{m} \varphi_i^{-1/\delta_i^*} \frac{\varepsilon_j z_i}{2\sigma_{j-1} t_{j-1}}$$

$$+ \sum_{i=0}^{m-2}\sum_{j=i+2}^{m} \varphi_i^{-1/\delta_i^*} \frac{\varepsilon_j z_i}{2\sigma_{j-2} t_j}$$

$$= \frac{\varphi_m^{-1/\delta_m^*}\varepsilon_m}{2\sigma_m t_m} z_m + \varphi_{m-1}^{-1/\delta_{m-1}^*}\xi_{m-1} z_{m-1}$$

$$+ \sum_{i=0}^{m-2} \varphi_i^{-1/\delta_i^*}\left(\xi_i + \sum_{j=i+2}^{m}\left(\frac{1}{2t_j}\Delta\left(\frac{1}{2\sigma_j}\right) - \frac{1}{t_{j-1}}\Delta\left(\frac{1}{\sigma_{j-1}}\right)\right)\right) z_i$$

$$= \sum_{i=0}^{m} f_{mi} z_i \qquad (y = \tilde{T}(u), z = E^{(\delta)}(y))$$

where

$$f_{mi} = \begin{cases} \varphi_i^{-1/\delta_i^*}\left(\xi_i + \sum_{j=i+2}^{m}\left(\frac{1}{2t_j}\Delta\left(\frac{1}{\sigma_j}\right) - \frac{1}{2t_{j-1}}\Delta\left(\frac{1}{\sigma_{j-1}}\right)\right)\right), & 0 \le i \le m-2 \\ \varphi_{m-1}^{-1/\delta_{m-1}^*}\xi_{m-1}, & i = m-1 \\ \frac{\varphi_m^{-1/\delta_m^*}\varepsilon_m}{2\sigma_m t_m} & i = m \\ 0, & i > m. \end{cases}$$

So, it is written that $z \in \{|T_\varphi|(\delta)\}^\beta$ if and only if $F = (f_{mi}) \in (\mathrm{l}(\delta), c)$. Using Lemma 1.1, we get that $\varepsilon \in \{|T_\varphi|(\delta)\}^\beta$ equals to $\varepsilon \in D_3 \cap D_5$ for $\delta_i > 1$ and $\varepsilon \in D_4 \cap D_5$ for $\delta_i \le 1$ for all i, which completes the part of the proof.

Because the proof of other parts can be proved in similar way, so it is left to reader.

Theorem 2.5. *Assume that (δ_i) is any bounded sequences of positive numbers for $i \in \mathbb{N}$. Also, $H = (h_{ni})$ be a matrix satisfying the following relation*

$$h_{ni} = \varphi_n^{1/\delta_n^*} \sum_{j=0}^{n} \Phi_{nj}\lambda_{ji}. \tag{5}$$

Then, $\Lambda \in (U, |T_\varphi|(\delta))$ iff $H \in (U, l(\delta))$.

Proof. Let take $u \in U$. Taking into consideration the equation (5), it is obtained immediately

$$\sum_{i=0}^{\infty} h_{ni}u_i = \varphi_n^{\frac{1}{\delta_n^*}} \sum_{j=0}^{n} \Phi_{nj} \sum_{i=0}^{\infty} \lambda_{ji}u_i\,,$$

and also $H_n(u) = (E^{(\delta)} \circ \tilde{T})_n(\Lambda(u))$ for $u \in U$. So, $\Lambda_n(u) \in |T_\varphi|(\delta)$ whenever $u \in U$ equals to $H_n(u) \in l(\delta)$ whenever $u \in U$. This concludes the proof of the theorem.

Theorem 2.6. *Assume that (δ_j) and (η_j) are bounded sequences of positive numbers with $\delta_j \leq 1$ and $\eta_j \geq 1$ and also $\hat{\Lambda}^{(\eta)} = E^{(\eta)} \circ \tilde{T} \circ \tilde{\Lambda}$. If $\Lambda \in \left(|T_\varphi|(\delta), |T_\psi|(\eta)\right)$, then Λ defines a bounded linear operator L_Λ such that $L_\Lambda(u) = \Lambda(u)$ for all $u \in |T_\theta|(\mu)$, and $\Lambda \in \left(|T_\varphi|(\delta), |T_\psi|(\eta)\right)$ if and only if there exists an integer $c > 1$ such that, for all n,*

$$\sum_{j=i+2}^{\infty} \lambda_{nj} \left(\frac{1}{2t_j}\Delta\left(\frac{1}{\sigma_j}\right) - \frac{1}{2t_{j-1}}\Delta\left(\frac{1}{\sigma_{j-1}}\right)\right) \text{exist for all } i, \tag{6}$$

$$\sup_{m,i}\left(\left|\varphi_i^{-1/\delta_i^*}\left(\xi_i^{(n)} + \sum_{j=i+2}^{m}\left(\frac{1}{2t_j}\Delta\left(\frac{1}{\sigma_j}\right) - \frac{1}{2t_{j-1}}\Delta\left(\frac{1}{\sigma_{j-1}}\right)\right)\right)\right|^{\delta_i} + \left|\varphi_{m-1}^{-1/\delta_{m-1}^*}\xi_{m-1}\right|^{\delta_{m-1}} + \left|\varphi_m^{-1/\delta_m^*}\frac{\lambda_{nm}}{2\sigma_m t_m}\right|^{\delta_m}\right) < \infty \tag{7}$$

$$\sup_i \sum_{n=0}^{\infty} \left|c^{-1/\delta_i}\hat{\lambda}_{ni}^{(\eta)}\right|^{\eta_n} < \infty. \tag{8}$$

Proof. Since $|T_\varphi|(\delta), |T_\psi|(\eta)$ are FK spaces, to prove the first part of the theorem it is sufficient to consider Theorem 4.2.8 of Wilasky **(1984)**, Λ determines a linear bounded operator L_Λ. To prove the second part of theorem consider $\Lambda \in \left(|T_\varphi|(\delta), |T_\psi|(\eta)\right)$. By Lemma 1.4, it is obtained immediately that $\Lambda \in \left(|T_\varphi|(\delta), |T_\psi|(\eta)\right)$ if and only if $\tilde{\Lambda} \in \left(l(\delta), |T_\psi|(\eta)\right)$ and $V^{(n)} \in (l(\delta), c)$ where

$$\tilde{\lambda}_{ni} = \varphi_i^{-1/\delta_i^*}\left(\lambda_{ni}\frac{1}{2\sigma_i t_i} + \left(\frac{1}{2\sigma_{i+1}t_{i+1}} - \frac{1}{2\sigma_i t_{i+1}} - \frac{1}{2\sigma_i t_i}\right)\lambda_{n,i+1} + \sum_{j=i+2}^{\infty}\lambda_{nj}\left(\frac{1}{2t_j}\Delta\left(\frac{1}{\sigma_j}\right) - \frac{1}{2t_{j-1}}\Delta\left(\frac{1}{\sigma_{j-1}}\right)\right)\right) n, i = 0,1,\ldots$$

$$v_{mi}^{(n)} = \begin{cases} \varphi_i^{-1/\delta_i^*}\left(\xi_i^{(n)} + \sum_{j=i+2}^{m}\left(\frac{1}{2t_j}\Delta\left(\frac{1}{\sigma_j}\right) - \frac{1}{2t_{j-1}}\Delta\left(\frac{1}{\sigma_{j-1}}\right)\right)\right), & 0 \le i \le m-2 \\ \varphi_{m-1}^{-1/\delta_{m-1}^*}\xi_{m-1}^{(n)}, & i = m-1 \\ \frac{\varphi_m^{-1/\delta_m^*}\lambda_{nm}}{2\sigma_m t_m} & i = m \\ 0, & i > m. \end{cases}$$

If we apply the Lemma 1.1 to the matrix $V^{(\mathrm{n})}$, we get the conditions (6) and (7). On the other hand, since $|T_\psi|(\eta) = \left(l(\eta)\right)_{E^{(\eta)}\circ\tilde{T}}$, for $u \in l(\delta)$, $\widetilde{\Lambda}(u) \in |T_\psi|(\eta)$ if and only if $\widehat{\Lambda}^{(\eta)}(u) = E^{(\eta)} \circ \tilde{T} \circ \widetilde{\Lambda}(u) \in l(\eta)$, and so $\widehat{\Lambda}^{(\eta)} \in \left(l(\delta), l(\eta)\right)$. Hence, the last statement gives us the condition (8) with Lemma 1.1 which concludes the proof.

Theorem 2.7 *Assume that* (δ_j) *is a bounded sequence of positive numbers with* $\delta_j > 1$ *and also* $\hat{\Lambda}^{(1)} = E^{(1)} \circ \tilde{T} \circ \tilde{\Lambda}$. *If* $\Lambda \in \left(|T_\varphi|(\delta), |T_\psi|\right)$, *then* Λ *defines a bounded linear operator* L_Λ *such that* $L_\Lambda(u) = \Lambda(u)$ *for all* $u \epsilon |T_\varphi|(\delta)$. *Also,* $\Lambda \in \left(|T_\varphi|(\delta), |T_\psi|\right)$ *if and only if there exists an integer* $c > 1$ *such that, for all* n,

$$\sum_{j=i+2}^{\infty}\lambda_{nj}\left(\frac{1}{2t_j}\Delta\left(\frac{1}{\sigma_j}\right) - \frac{1}{2t_{j-1}}\Delta\left(\frac{1}{\sigma_{j-1}}\right)\right) exist\ for\ all\ i, \tag{9}$$

$$\sup_m\left(\sum_{i=0}^{m-2}\frac{c^{-1/\delta_i^*}}{\varphi_i}\left|\xi_i^{(n)} + \sum_{j=i+2}^{m}\lambda_{nj}\left(\frac{1}{2t_j}\Delta\left(\frac{1}{\sigma_j}\right) - \frac{1}{2t_{j-1}}\Delta\left(\frac{1}{\sigma_{j-1}}\right)\right)\right|^{\delta_i^*} + \frac{c^{-1/\mu_{m-1}^*}}{\varphi_{m-1}}\left|\xi_{m-1}^{(n)}\right|^{\delta_{m-1}^*} + \frac{c^{-1/\delta_m^*}}{\varphi_m}\left|\frac{\lambda_{nm}}{2\sigma_m t_m}\right|^{\delta_m^*}\right) < \infty \tag{10}$$

$$\sum_{i=0}^{\infty}\left(\sum_{n=0}^{\infty}\left|c^{-1}\hat{\lambda}_{ni}^{(1)}\right|\right)^{\delta_i^*} < \infty \tag{11}$$

Proof. The first part of the theorem can be proved in a similar way as above, so it is left to reader. Since $|T_\varphi|(\delta) = (l(\delta))_{E^{(\delta)}\circ\tilde{T}}$, it follows from Lemma 1.4, $\Lambda \in (|T_\varphi|(\delta), |T_\psi|)$ if and only if $\widetilde{\Lambda} \in (l(\delta), |T_\psi|)$ and $V^{(n)} \in (l(\delta), c)$ where the matrices $\widetilde{\Lambda}$ and $V^{(\mathrm{n})}$ defined as in Theorem 2.6. Applying Lemma 1.1 to $V^{(\mathrm{n})}$, we get immediately the conditions (9) and (10). On the other hand, since $|T_\psi| = (l)_{E^{(1)}\circ\tilde{T}}$, $\widetilde{\Lambda} \in (l(\delta), |T_\psi|)$ equals to $\widehat{\Lambda}^{(1)} = E^{(1)} \circ \tilde{T} \circ \widetilde{\Lambda} \in (l(\delta), l)$. Again, applying Lemma 1.1 to $\widehat{\Lambda}^{(1)}$, the condition (11) is obtained, and so the proof is comleted.

Acknowledgement: The author would like to thank the referees for their valuable comments and fruitful suggestions that improved the readability of the manuscript.

Availability of Data and Materials: No data were used to support the findings of the study.

Conflicts of Interest: The author declares no conflicts of interest.

Funding: There was no funding support for this study.

Authors' Contributions: The author contributed solely to this work. The author has read and approved the final version of the manuscript.

References:

Dağlı, M.C., & Yaying, T. (2023). Some new paranormed sequence spaces derived by regular Tribonacci matrix. The Journal of Analysis, 31(1), 109-127.

Gökçe, F., (2021). Compact and Matrix Operators on the Space $|\bar{N}_P^\theta|_k$. Fundamental Journal of Mathematics and Applications 4 (2), 124-133.

Gökçe, F. (2022). Compact matrix operators on Banach space of absolutely k-summable series. Turkish Journal of Mathematics, 46(3), 1004-1019.

Gökçe, F. (2025). On absolute Tribonacci series spaces and some matrix operators. Mathematical Sciences and Applications E-Notes, 13(1), 1-11.

Gökçe, F., & Sarıgöl, M.A. (2020). Some matrix and compact operators of the absolute Fibonacci series spaces. Kragujevac Journal of Mathematics, 44 (2), 273–286,

Gökçe, F., & Sarıgöl, M.A. (2020a). On absolute Euler spaces and related matrix operators. Proceedings of the National Academy of Sciences, India. Section A. Physical Sciences, 90(5), 769-775

Gökçe, F., & Sarıgöl, M.A. (2020b). Series spaces derived from absolute Fibonacci summability and matrix transformations. Bollettino dell'Unione Matematica Italiana, 13, 29-38

Gökçe, F., & Sarıgöl, M.A. (2019). Generalization of the absolute Cesaro space and some matrix transformations. Numerical Functional Analysis and Optimization , 40, 1039-1052

Gökçe, F., & Sarıgöl, M.A. (2019a). Extension of Maddox's space $l(\mu)$ with Nörlund means. Asian-European Journal of Mathematics, 12(6), 2040005.

Gökçe, F., & Sarıgöl, M.A. (2018). Generalization of the space $l(p)$ derived by absolute Euler summability and matrix operators. Journal of Inequalities and Applications, 2018, 133.

Gökçe, F., & Sarıgöl, M.A. (2018a). A new series space $\left|\bar{N}_p^\theta\right|(\mu)$ and matrix transformations with applications. Kuwait Journal of Science, 45 (4), 1-8.

Grosse-Erdmann, K.G. (1993). Matrix transformations between the sequence spaces of Maddox. Journal of Mathematical Analysis and Applications, 180, 223-238

Maddox, I. J. (1967). Spaces of strongly summable sequences. The Quarterly Journal of Mathematics, 18, 345-355.

Maddox, I. J. (1968). Paranormed sequence spaces generated by infinite matrices. Mathematical Proceedings of the Cambridge Philosophical Society, 64, 335-340.

Maddox, I. J. (1969). Some properties of paranormed sequence Spaces. Journal of the London Mathematical Society. Second Series, 316-322.

Malkowsky, E., Rakocevic, V. (2000). An introduction into the theory of sequence space and measures of noncompactness. Zbornik. Radova (Beogr), 9 (17), 143-234.

Malkowsky, E., Rakocevic V. (2007). On matrix domains of triangles. Applied Mathematics Computations. 189 (2), 1146-1163.

Sarıgöl, M.A. (2010). On the local properties of factored Fourier series. Applied Mathematics Computations 216 (11), 3386-3390

Sarıgöl, M.A. (2013). An inequality for matrix operators and its applications. Journal of Classical Analysis, 2, 145-150.

Wilansky, A., (1984). Summability through Functional Analysis. Mathematical Studies 85, North–Holland, Amsterdam.

Yaying, T., & Hazarika, B. (2020). On sequence spaces defined by the domain of a regular Tribonacci matrix. Mathematica Slovaca, 70 (3), 697-706.

Yaying, T., & Kara, M.I. (2021). On sequence spaces defined by the domain of tribonacci matrix in c_0 and c. Korean Journal of Mathematics, 29 (1), 25-40.