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Research Article

Exploring the stability of two different forms of a same functional equation

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Abstract: Within this manuscript, we shall delve into the direct method and explore the general solution as well as the classical stability of the ensuing reciprocal functional equation

$$[f(x-y+z)-f(3x+y-z)][f(x+y-z)+2f(x-y+z)]-2f(x-y+z)^2=0$$

in Banach spaces. Moreover, another objective of this work is to obtain another interesting result of stability of the same equation but this time, we make a little change of the above equation and we obtain a new stability result however with a supplementary condition on the mapping f.

Keywords: Reciprocal functional Equation, Hyers-Ulam-Rassias Stability, Banach Space, General Solutions, Direct Method.

MSC Subject Classification: 39B52, 39B72.

1. Introduction

Stability of the functional equations, difference and differential equations has been a hot topic over the last decades. In the first instance, it propped the question of Ulam **(1960)**. In his famous lecture in 1940, at the Mathematics Club of University of Wisconsin. The question of Ulam is whether it is true, that the solution of an equation differing slightly from a given one, should be close to the solution of the given equation? Hyers **(1941)** gave a partially response in the case of linear functional equations in Banach spaces. After that, a large number of papers have been published in connection with various generalizations of Ulam's problem and Hyers theorem, see for example the books (**(Aczel & Dhombres, 1989; Czerwik, 2002; Rassias & Brdzek, 2012; Rassias, 2010, 2014; Sahoo & Kannappan, 2011)**) and papers (**(Benzarouala et al., 2023; Kattan & Hammad, 2023; Sadani, 2020, 2022, 2023; Sayyari et al., 2024; Santra et al., 2023; Tamilvanan et al., 2024)**).

The present paper has two objectives, the first is to introduce and to obtain the solutions and stability of the following new reciprocal functional equation

$$[f(x-y+z)-f(3x+y-z)][f(x+y-z)+2f(x-y+z)]-2f(x-y+z)^2=0 \qquad (1.1)$$

for all $x,y,z\in X$, such that X denote the set of real numbers, and Y a Banach space. The second objective is to modify the above equation by dividing by the term $f(x+y-z)+2f(x-y+z)$, which is assumed to be non-zero for all $x,y,z\in X$ to obtain the following new functional equation

$$f(x-y+z)-f(3x+y-z)-\frac{2f(x-y+z)^2}{f(x+y-z)+2f(x-y+z)}=0 \qquad (1.2)$$

The general solution of (1.2) is the same as (1.1), but the essential question is that the stability conditions imposed on the first equation are the same as those of the second?– The answer is negative. We will see that the conditions imposed are not the same. Before answering this question in detail, our initial focus shall be on seeking a general solution to the equation at hand (1.1).

2. General solutions of (1.1)

Within this section, we aim to derive a generalized solution to the functional equation (1.1).

Theorem 2.1. *Consider a mapping f from X to Y, which satisfies the given functional equation* (1.1) *such that* $f(x)\neq 0,\forall x\in X$. *Then, f is a reciprocal mapping.*

Proof. Suppose that $f(x)\neq 0,\forall x\in X$. Upon substituting (x,y,z) with $(x,0,0)$ in equation (1.1), the resulting expression is as follows

$$f(x)(f(x)-3f(3x))=0,\forall x\in X \qquad (2.1)$$

This implies that

$$f(3x)=\frac{1}{3}f(x),\forall x\in X \qquad (2.2)$$

Upon substituting (x,y,z) with $(x,-x,x)$ in (1.1), we get

$$(-f(x)+f(3x))f(-x)-2f(x)f(3x)=0,\forall x\in X \qquad (2.3)$$

Now, using (2.2), we obtain

$$\frac{-2}{3}f(x)(f(-x)+f(x))=0$$

and since $f(x)\neq 0,\forall x\in X$, we get

$$f(-x)=-f(x),\forall x\in X \qquad (2.4)$$

then, f is odd. Upon substituting (x,y,z) with $(-x,-x,2x)$ in (1.1) and applying (2.4) and (2.2), we can derive the following expression

$$f(2x)=\frac{1}{2}f(x), \forall x\in X \tag{2.5}$$

After substituting (x,y,z) with $(2x,x,2x)$ in (1.1), and utilizing the fact that f is an odd function, we can derive the following expression

$$f(5x)=\frac{1}{5}f(x), \forall x\in X.$$

We see that for $k=2,3,4,5$, we have

$$f(kx)=\frac{1}{k}f(x), \forall x\in X \tag{2.6}$$

Assuming that (2.6) holds true for $n=k$, we can prove that it also holds true for $n=k+1$ by substituting $(x,y,z)=\left(\frac{k}{2}x,x,\frac{k}{2}x\right)$ and obtaining the following expression

$$(f(x)-2f(x(k+1)))f(x(k-1))-f(x(k+1))f(x)=0, \forall x\in X$$

and using (2.6), we get

$$f((k+1)x)=\frac{1}{k+1}f(x), \forall x\in X.$$

Therefore, we can conclude that f is a reciprocal mapping.

3. Stability result of equation (1.1)

Our focus now is to investigate the generalized Hyers–Ulam stability of the functional equation (1.1) using the direct method. To begin, we introduce the following notation

$$Rf(x,y,z,t)=[f(x-y+z)-f(3x+y-z)][f(x+y-z)+$$
$$+2f(x-y+z)]-2f(x-y+z)^2$$

Theorem 3.1. *Let $\phi: X\to\mathbb{R}_+$ be a mapping satisfying*

$$\Phi(x,y,z)=\theta\sum_{k=0}^{\infty}\parallel 3^k x\parallel^p\, 3^k\phi(3^kx,3^ky,3^kz)<\infty$$

and

$$\lim_{k\to\infty}\theta\parallel 3^k x\parallel^p\, 3^k\phi(3^kx,3^ky,3^kz)=0$$

with $p<-2$ and $\theta>0$. Assuming that a mapping $f: X\to Y$ satisfies the following conditions:

1. $\|Rf(x,y,z)\|\leq\parallel x\parallel^p\,\phi(x,y,z)$ *for all $x,y,z,t\in X$,*
2. $f(x)\neq 0$ *and* $\parallel f(x)\parallel>\theta^{-1}\parallel x\parallel^{-p}, \forall x\in X^*$ *with $\theta>0$ and $p<-2$.*

Then f is bounded and there exists a unique reciprocal mapping $g: X\to Y$ such that

$$\|f(x) - g(x)\| \le \Phi(x, x, x) \tag{3.1}$$

for all $x \in X$. The mapping g is defined as

$$g(x) = \lim_{n\to\infty} 3^n f(3^n x), \forall x \in X.$$

Proof. Firstly, upon substituting (x, y, z) with $(x, 0, x)$ in (2), we derive the following

$$\| 2f(x)^2 \| \le \| x \|^p \ \phi(x, 0, x)$$

This implies that

$$\| f(x) \| \le \| x \|^{p/2} \sqrt{\frac{\phi(x, 0, x)}{2}}$$

then, f is bounded. Secondly, upon replacing (x, y, z) with (x, x, x) in (2), we get

$$\| f(x)\big(f(x) - 3f(3x)\big) \| \le \phi(x, x, x), \forall x \in X.$$

Using the condition (2), we have

$$\| f(x) - 3f(3x) \| \le \frac{\phi(x,x,x)}{\|f(x)\|} \le \phi(x, x, x)\theta \| x \|^p \tag{3.2}$$

for all $x \in X$. Replacing x by $3x$ in (3.2) and multiplying by 3 and summing with (3.2), we get

$$\| f(x) - 3^2 f(9x) \| \le \phi(x, x, x)\theta \| x \|^p + 3^{p+1}\phi(3x, 3x, 3x)\theta \| x \|^p \tag{3.3}$$

Repeating the process, we have

$$\| f(x) - 3^n f(3^n x) \| \le \theta \| x \|^p \sum_{k=0}^{n-1} 3^{k(p+1)}\phi(3^k x, 3^k x, 3^k x) \tag{3.4}$$

for all $x \in X$ and any positive integer n. In order to establish the convergence of the sequence $3^n f(3^n x)$, we multiply inequality (3.4) by 3^m and replace x by $3^m x$ to find that for $n, m > 0$

$$\|3^m f(3^m) - 3^{n+m} f(3^{n+m})\| = 3^m\|f(3^m) - 3^n f(3^{n+m})\| \le$$

$$\le \theta \| x \|^p \sum_{k=0}^{n-1} 3^{(k+m)(p+1)}\phi(3^{k+m}x, 3^{k+m}x, 3^{k+m}x) \tag{3.5}$$

Taking limit $m \to \infty$ in (3.5), It is apparent that the right-hand side of the inequality (3.5) tends to 0, which indicates that the sequence $\{3^n f(3^n x)\}$ is a Cauchy sequence. Hence, we can define a mapping

$$g(x) = \lim_{n\to\infty} 3^n f(3^n x), \forall x \in X$$

By letting $n \to \infty$ in (3.4), we arrive at the formula (3.1).

To prove the uniqueness of the function g, let us assume that there exists a function $g' : X \to Y$ which satisfies (1.1) and the inequality (3.1). Hence it follows from (3.1) that

$$\|g'(x) - g(x)\| = 3^n\|g'(3^n x) - g(3^n x)\| \le 3^n(\|g'(3^n x) - f(3^n x)\|$$

$$+3^n\|f(3^n x) - g(3^n x)\|) \le 2 \| x \|^p \sum_{k=0}^{n-1} 3^{(k+2n)(p+1)}\phi(3^{n+k}x, 3^{k+n}x, 3^{n+k}x) \tag{3.6}$$

for all $x \in X$. By letting $n \to \infty$ in the previous inequality, we obtain that g is unique.

Finally, to prove that g satisfies (1.1), replacing (x, y, z) by $(3^k x, 3^k y, 3^k z)$ and multiplying by 3^{2k} in (2), we obtain that

$$3^{2k}\|Rf(3^k x, 3^k y, 3^k z)\| \leq 3^{k(p+2)} \parallel x \parallel^p \phi(3^k x, 3^k y, 3^k z)$$

for all $x, y, z \in X$. Letting $k \to \infty$ in the above inequality and using the definition of $g(x)$, we see that

$$[f(x-y+z) - f(3x+y-z)][f(x+y-z) + 2f(x-y+z)] - 2f(x-y+z)^2 = 0.$$

This completes the proof of the theorem.

4. Stability of the modified functional equation of (1.1).

Our current focus is on examining the generalized Hyers-Ulam stability of the modified functional equation (1.2) from (1.1). To do this, we set the following notation

$$Tf(x, y, z) = f(x-y+z) - f(3x+y-z) - \frac{2f(x-y+z)^2}{f(x+y-z)+2f(x-y+z)}$$

such that $f(x+y-z) + 2f(x-y+z) \neq 0, \forall x, y, z \in X$.

Theorem 4.1. *Let $\phi: X \times X \times X \to \mathbb{R}_+$ be a mapping satisfying*

$$\Phi(x, y, z) = \sum_{k=0}^{\infty} 3^{k+1} \phi(3^k x, 3^k y, 3^k z) < \infty \tag{4.1}$$

and

$$\lim_{k\to\infty} 3^{k+1} \phi(3^k x, 3^k y, 3^k z) = 0 \tag{4.2}$$

for all $x, y, z \in X$. Assume that $f: X^* \to Y$ is a mapping satisfying

$$\|Tf(x, y, z)\| \leq \phi(x, y, z) \tag{4.3}$$

for all $x, y, z \in X$. Then, there exists a unique reciprocal mapping $g: X \to Y$ which satisfy the functional equation (1.1) *and inequality*

$$\|f(x) - g(x)\| \leq \Phi(x, x, x). \tag{4.4}$$

The mapping g is defined as

$$g(x) = \lim_{n\to\infty} 3^n f(3^n x), \forall x \in X.$$

Proof. To prove this, we substitute (x, y, z) with (x, x, x) in equation (4.3), and then use similar reasoning as in Theorem 3.1.

Theorem 4.2.: *Let $\phi: X \times X \times X \to \mathbb{R}_+$ be a mapping satisfying*

$$\Phi(x, y, z) = \sum_{k=1}^{\infty} \frac{1}{3^{k-1}} \phi\left(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k}\right) < \infty \tag{4.5}$$

and

$$\lim_{k\to\infty} \frac{1}{3^{k-1}} \phi\left(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k}\right) = 0 \tag{4.6}$$

for all $x, y, z \in X$. Assume that $f: X^* \to Y$ is a mapping satisfying

$$\|Tf(x,y,z)\| \le \phi(x,y,z) \qquad (4.7)$$

for all $x,y,z \in X$. *Then, there exists a unique reciprocal mapping* $h: X \to Y$ *which satisfy the functional equation* (1.1) *and inequality*

$$\|f(x) - h(x)\| \le \Phi(x,x,x). \qquad (4.8)$$

The mapping g *is defined as*

$$h(x) = \lim_{n\to\infty} 3^{-n} f(3^{-n}x), \forall x \in X.$$

Proof. To prove this, we substitute (x,y,z) with $(x/3, x/3, x/3)$ in equation (4.7), and then use similar reasoning as in Theorem 4.1.

The following corollaries are an immediate consequence of Theorem 4.1 and Theorem 4.2.

Corollary 4.3. *Suppose that the mapping* f *from* X *to* Y *satisfies the functional inequality*

$$\|Tf(x,y,z)\| \le \theta(\| x \|^p \| y \|^q \| z \|^r)$$

for all $x,y,z \in X$ *with* $p+q+r < -1$, *then there exists a unique reciprocal mapping* $g: X \to Y$ *such that*

$$\|f(x) - g(x)\| \le \frac{3\theta}{1 - 3^{p+q+r+1}} \| x \|^{p+q+r}, \forall x \in X.$$

Corollary 4.4. *If a mapping* $f: X \to Y$ *satisfies the functional inequality*

$$\|Tf(x,y,z)\| \le \theta(\| x \|^p + \| y \|^p + \| z \|^p)$$

for all $x,y,z \in X$ *with* $p < -1$, *then there exists a unique reciprocal mapping* $g: X^* \to Y$ *such that*

$$\|f(x) - g(x)\| \le \frac{9\theta}{1 - 3^{p+1}} \| x \|^p$$

for all $x \in X$.

Corollary 4.5. *Suppose* ε *is a non-negative real number. If a function* $f: X \to Y$ *satisfies the following functional inequality*

$$\|Tf(x,y,z)\| \le \varepsilon$$

for all $x,y,z \in X$, *then there exists a unique reciprocal mapping* $g: X \to Y$ *such that*

$$\|f(x) - h(x)\| \le \tfrac{3}{2}\varepsilon.$$

for all $x \in X$.

Corollary 4.6. *If a mapping* $f: X \to Y$ *satisfies the functional inequality*

$$\|Tf(x,y,z)\| \le \theta(\| x \|^p + \| y \|^p + \| z \|^p)$$

for all $x, y, z \in X$ *with* $p > -1$, *then there exists a unique reciprocal mapping* $g: X \to Y$ *such that*

$$\|f(x) - h(x)\| \leq \frac{9\theta}{3^{p+1}-1} \| x \|^p$$

for all $x \in X$.

Corollary 4.7. *If a mapping* $f: X \to Y$ *satisfies the functional inequality*

$$\|Tf(x, y, z)\| \leq \theta(\| x \|^p \| y \|^q \| z \|^r)$$

for all $x, y, z \in X$ with $p + q + r > -1$, then there exists a unique reciprocal mapping $g: X^* \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{3\theta}{3^{p+q+r+1}-1} \| x \|^{p+q+r}$$

for all $x \in X$.

5. Conclusion

We have demonstrated that the manner in which a functional equation is expressed affects the requirements for its stability in the Hyer-Ulam-Rassias sense (see Theorem 3.1 and Theorem 4.1).

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