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Research Article

On convergence in G_Q metric spaces

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Abstract: This paper reviews some basic aspects and proposes and explores the idea of convergence for triple sequences in G_Q-metric space. Furthermore, a thorough analysis and definition of statistical convergence in this context are provided. This link is explored and its ramifications discussed in the last part, which focuses on the relationship between strong summability and the statistical convergence of G_Q-metric spaces.

Keywords: Statistical convergence, G_Q-metric space, Cauchy sequence, Quaternion space, Triple sequence

MSC 2020: 40A35, 40A05, 54E35.

1. Introduction

For many years, one of the most important and active areas of research in pure mathematics has been the study of sequence convergence and summability theory. Moreover, it has made significant contributions to a wide range of fields, including computer science, mathematical modelling, functional analysis, topology, measure theory, and applied mathematics. In recent years, the idea of statistical convergence of sequences has been applied widely in mathematics. Fast **(Fast, 1951)** conducted a first exploration of statistical convergence. Since then, a number of mathematicians have investigated the statistical convergence and convergence features and applied these ideas to many disciplines. Further references can be found in **(Connor, 1988)**, **(Fridy, 1985)**, **(Gürdal, 2004)**, **(Kandemir et al., 2022)**, **(Kişi, Çakal & Gürdal, 2024a;2024b;2024c)**, **(Kolancı, Gürdal & Kişi, 2024)**, **(Nabiev, Savaş & Gürdal, 2019)**, **(Şahiner, Gürdal & Düden, 2007)**, **(Temizsu & Et, 2023)**, **(Tripathy & Et, 2005)**.

A distance function, or metric, extends the concept of physical distance in mathematical analysis. The concept of a metric space was expanded and generalized by several writers to include complex and quaternion metric spaces. Following Mustafa and Sims' presentation of the idea of generalized metric spaces (also known as G −metric spaces) in **(Mustafa & Sims, 2006)**, a number of studies were written expanding on this idea to include complex generalized metric spaces. Metrics in this domain indicate the separation of three locations.

Quaternions are a number system that goes beyond complex numbers. Hamilton, an Irish mathematician, first proposed it in 1843 to explain three-dimensional mechanics. The noncommutative multiplication of two quaternions is one property of quaternions. Quaternion analysis is covered in detail in **(Azam, Fisher & Khan, 2011)** and its references. This paper proposes a type of convergence types in quaternion-valued G −metric spaces (or $G_{\mathbf{Q}}$ −metric spaces) to extend the G −metric spaces introduced by **(Mustafa & Sims, 2006)**, $G_{\mathbf{Q}}$ −metric spaces introduced by **(Al-Ahmadi, 2017)**, and various forms of convergence well known in the literature. The motivation behind this proposal stems from the practical applications of quaternions.

This paper reviews fundamental aspects and proposes the concept of convergence for triple sequences in $G_{\mathbf{Q}}$ −metric space, alongside a detailed analysis and definition of statistical convergence in this context. The implications of these results are particularly relevant for fields such as numerical analysis, optimization, and data science, where convergence properties play a crucial role in algorithms and their efficiency. Additionally, the relationship between strong summability and statistical convergence in $G_{\mathbf{Q}}$ −metric spaces offers insights for further applications in solving problems related to fixed-point theory and dynamical systems. In this study, we also examine convergence and statistical convergence for triple sequences in quaternion-valued G-metric spaces and discuss some of their basic properties. The paper concludes with an exploration of the complex relationship between statistical convergence in $G_{\mathbf{Q}}$ −metric spaces and the concept of strong summability, highlighting its potential applications and making the findings accessible to a broader scientific community, while still being relevant for those with advanced mathematical expertise.

2. Preliminaries

Let us review the meanings of statistical convergence, p −strongly Cesàro summability, and natural density (details may be found in the above-mentioned texts).

The asymptotic (or natural) density for a set S of positive integers is defined as follows:

$$\delta(S) := \lim_{s} \frac{1}{s} |\{u \leq s : u \in S\}|.$$

A sequence (w_u) is said to be statistically convergent to w, if for every $\varepsilon > 0$

$$\lim_s \frac{1}{s}|\{u \leq s: |w_u - w| \geq \varepsilon\}| = 0.$$

A sequence (w_u) is said to be p −strongly Cesàro summable to w, if

$$\lim_s \frac{1}{s}\sum_{u=1}^{s} |w_u - w|^p = 0.$$

Now, let us review some fundamental ideas about quaternion spaces (see, for instance, **(Farenick & Pidkowich, 2003)**, **(Omran & Al-Harthy, 2011)**). One dimensional real algebra in four dimensions is called the space of quaternions, or $\mathbf{Q}$. The multiplicative identity of $\mathbf{Q}$ is represented by $\mathbf{1_Q}$, whereas the null element of $\mathbf{Q}$ is denoted by $\mathbf{0_Q}$. Three so-called imaginary units (i, j, and k) are included in the space $\mathbf{Q}$. By definition, they satisfy:

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, kj = -jk = i \text{ and } ki = -ik = j.$$

Assuming that $1, i, j, k$ constitute a real vector basis of $\mathbf{Q}$, every $q_0 \in \mathbf{Q}$ may be expressed as $q_0 = x_0 + x_1 i + x_2 j + x_3 k$, where x_0, x_1, x_2, and x_3 are elements of $\mathbb{R}$.

It was noted in **(El-Sayed, Omran & Asad, 2014)** if any of the following circumstances holds true, $q_1 \preccurlyeq q_2$.

(i). $\mathrm{Re}(q_1) = \mathrm{Re}(q_2); \mathrm{Im}_{s_1}(q_1) = \mathrm{Im}_{s_1}(q_2);$ where $s_1 = j, k; \mathrm{Im}_i(q_1) = \mathrm{Im}(q_2);$

(ii). $\mathrm{Re}(q_1) = \mathrm{Re}(q_2); \mathrm{Im}_{s_2}(q_1) = \mathrm{Im}_{s_2}(q_2);$ where $s_2 = i, k; \mathrm{Im}_j(q_1) = \mathrm{Im}_j(q_2);$

(iii). $\mathrm{Re}(q_1) = \mathrm{Re}(q_2); \mathrm{Im}_{s_3}(q_1) = \mathrm{Im}_{s_3}(q_2);$ where $s_3 = i, j; \mathrm{Im}_k(q_1) = \mathrm{Im}_k(q_2);$

(iv). $\mathrm{Re}(q_1) = \mathrm{Re}(q_2); \mathrm{Im}_{s_1}(q_1) < \mathrm{Im}_{s_1}(q_2); \mathrm{Im}_i(q_1) = \mathrm{Im}(q_2);$

(v). $\mathrm{Re}(q_1) = \mathrm{Re}(q_2); \mathrm{Im}_{s_2}(q_1) < \mathrm{Im}_{s_2}(q_2); \mathrm{Im}\, m_j(q_1) = \mathrm{Im}\, m_j(q_2);$

(vi). $\mathrm{Re}(q_1) = \mathrm{Re}(q_2); \mathrm{Im}_{s_3}(q_1) < \mathrm{Im}_{s_3}(q_2); \mathrm{Im}\, m_k(q_1) = \mathrm{Im}\, m_k(q_2);$

(vii). $\mathrm{Re}(q_1) = \mathrm{Re}(q_2); \mathrm{Im}_s(q_1) < \mathrm{Im}_s(q_2);$

(viii). $\mathrm{Re}(q_1) < \mathrm{Re}(q_2); \mathrm{Im}_s(q_1) = \mathrm{Im}_s(q_2);$

(ix). $\mathrm{Re}(q_1) < \mathrm{Re}(q_2); \mathrm{Im}_{s_1}(q_1) = \mathrm{Im}_{s_1}(q_2);$ where $s_1 = j, k; \mathrm{Im}(q_1) = \mathrm{Im}(q_2);$

(x). $\mathrm{Re}(q_1) < \mathrm{Re}(q_2); \mathrm{Im}_{s_2}(q_1) = \mathrm{Im}_{s_2}(q_2); \mathrm{Im}_j(q_1) < \mathrm{Im}_j(q_2);$

(xi). $\mathrm{Re}(q_1) < \mathrm{Re}(q_2); \mathrm{Im}_{s_3}(q_1) = \mathrm{Im}_{s_3}(q_2); \mathrm{Im}\, m_k(q_1) < \mathrm{Im}_k(q_2);$

(xii). $\mathrm{Re}(q_1) < \mathrm{Re}(q_2); \mathrm{Im}_{s_1}(q_1) < \mathrm{Im}_{s_1}(q_2); \mathrm{Im}_i(q_1) = \mathrm{Im}_i(q_2);$

(xiii). $\mathrm{Re}(q_1) < \mathrm{Re}(q_2); \mathrm{Im}_{s_2}(q_1) < \mathrm{Im}_{s_2}(q_2); \mathrm{Im}_j(q_1) = \mathrm{Im}_j(q_2);$

(xiv). $\mathrm{Re}(q_1) < \mathrm{Re}(q_2); \mathrm{Im}_{s_3}(q_1) < \mathrm{Im}_{s_3}(q_2); \mathrm{Im}\, k(q_1) = \mathrm{Im}\, k(q_2);$

(xv). $\mathrm{Re}(q_1) < \mathrm{Re}(q_2)\,; \mathrm{Im}_s(q_1) < \mathrm{Im}_s(q_2)$

(xvi). $\mathrm{Re}(q_1) = \mathrm{Re}(q_2); \mathrm{Im}_s(q_1) = \mathrm{Im}_s(q_2).$

Remark 2.1. In particular, we note that $q_1 \precsim q_2$ if $q_1 \neq q_2$ and one from (i) to (xvi) is satisfied. Also, we note that $q_1 \prec q_2$ if only (xv) is satisfied. It should be remarked that

$$q_1 \preccurlyeq q_2 \Rightarrow |q_1| \leq |q_2|.$$

Definition 2.2. **(Al-Ahmadi, 2017)** *Let $\digamma$ be an non-empty set and let $G_{\boldsymbol{Q}}: \digamma \times \digamma \times \digamma \to \boldsymbol{Q}$ be a function satisfying the following conditions*

1. $G_{\boldsymbol{Q}}(x,y,z) = \mathbf{0}_{\boldsymbol{Q}}$ *if* $x = y = z$.
2. $\mathbf{0}_{\boldsymbol{Q}} \prec G_{\boldsymbol{Q}}(x,x,y)$ *for all* $x, y \in \digamma$ *with* $x \neq y$.
3. $G_{\boldsymbol{Q}}(x,x,y) \preccurlyeq G_{\boldsymbol{Q}}(x,y,z)$ *for all* $x,y,z \in \digamma$ *with* $y \neq z$.
4. $G_{\boldsymbol{Q}}(x,y,z) = G_{\boldsymbol{Q}}(x,z,y) = G_{\boldsymbol{Q}}(y,z,x) = \cdots$
5. $G_{\boldsymbol{Q}}(x,y,z) \preccurlyeq G_{\boldsymbol{Q}}(x,a,a) + G_{\boldsymbol{Q}}(a,y,z)$ *for all* $x,y,z,a \in \digamma$.

Then the function $G_{\boldsymbol{Q}}$ is called a quaternion-valued generalized metric or, more specifically, a quaternion-valued $G_{\boldsymbol{Q}}$ −metric on $\digamma$ and the pair $(\digamma, G_{\boldsymbol{Q}})$ is called a quaternion-valued G −metric space.

Example 2.3. Let $\digamma = \mathbf{Q}$ be a set of quaternion number. Define a quaternion valued function $G_{\mathbf{Q}}: \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q} \to \mathbf{Q}$ by

$$\begin{aligned} &G_{\mathbf{Q}}(q_1, q_2, q_3) \\ &= |x_0^1 - x_0^2| + |x_0^1 - x_0^3| + |x_0^2 - x_2^3| + i(|x_1^1 - x_1^2| + |x_1^1 - x_1^3| + |x_1^2 - x_1^3|) \\ &+ j(|x_2^1 - x_2^2| + |x_2^1 - x_2^3| + |x_2^2 - x_2^3|) + k(|x_3^1 - x_3^2| + |x_3^1 - x_3^3| + |x_3^2 - x_3^3|), \end{aligned}$$

where $q_r = x_0^r + x_1^r i + x_2^r j + x_3^r k$ for $r = 1,2,3$. Then $(\mathbf{Q}, G_{\mathbf{Q}})$ is a quaternion valued G −metric space.

Proposition 2.4. **(Al-Ahmadi, 2017)** *Let $(\digamma, G_{\boldsymbol{Q}})$ be a quaternion valued G −metric space. Then for all x, y, z and $a \in \digamma$ the following properties hold.*

(1). $G_{\mathbf{Q}}(x,y,z)=0 \Rightarrow x=y=z$

(2). $G_{\mathbf{Q}}(x,y,z) \preccurlyeq G_{\mathbf{Q}}(x,x,y)+G_{\mathbf{Q}}(x,x,z)$

(3). $G_{\mathbf{Q}}(x,y,y) \preccurlyeq 2G_{\mathbf{Q}}(y,x,x)$

(4). $G_{\mathbf{Q}}(x,y,z) \preccurlyeq G_{\mathbf{Q}}(x,a,z)+G_{\mathbf{Q}}(a,y,z)$

(5). $G_{\mathbf{Q}}(x,y,z) \preccurlyeq \frac{2}{3}\left(G_{\mathbf{Q}}(x,y,a)+G_{\mathbf{Q}}(x,a,z)+G_{\mathbf{Q}}(a,y,z)\right)$

(6). $G_{\mathbf{Q}}(x,y,z) \preccurlyeq 2\left(G_{\mathbf{Q}}(x,x,a)+G_{\mathbf{Q}}(y,y,a)+G_{\mathbf{Q}}(z,z,a)\right)$.

3. Main Results

In this section we define some types of convergence in $G_{\mathbf{Q}}$ −metric space and discuss some of its basic properties.

Definition 3.1. *Let $(\digamma, G_Q)$ be a G_Q −metric space, and let (w_{nml}) be a triple sequence of points of $\digamma$. We say that (w_{nml}) is G_Q −convergent to $w \in \digamma$ if*

$$\forall q_0 \in \boldsymbol{Q}: 0 \prec q_0, \exists n_0 \in \mathbb{N} \Rightarrow G_Q\left(w, w_{n_1m_1l_1}, w_{n_2m_2l_2}\right) \prec q_0, \forall n_1,n_2,m_1,m_2,l_1,l_2 \geq n_0$$

We call w the limit of the triple sequence w_{nml} and write $w_{nml} \to_{G_Q} w$ or $G_Q - \lim_{n,m,l\to\infty} w_{nml} = w$.

Definition 3.2. *Let $(\digamma, G_Q)$ be a G_Q −metric space. The triple sequence (w_{nml}) is said to be a G_Q −Cauchy sequence if, for every $q_0 \in \boldsymbol{Q}, 0 \prec q_0$, there is a positive integer k such that*

$$\boldsymbol{G_Q}\left(w_{n_0m_0l_0}, w_{n_1m_1l_1}, w_{n_2m_2l_2}\right) \prec q_0$$

for all $n_0,n_1,n_2,m_0,m_1,m_2,l_0,l_1,l_2 \geq k$.

Definition 3.3. *A G_Q −metric space $(\digamma, G_Q)$ is said to be quaternion valued complete if every G_Q −Cauchy triple sequence is G_Q −convergent in $(\digamma, G_Q)$.*

Proposition 3.4. *Let $(\digamma, G_Q)$ be a G_Q −metric space and let (w_{nml}) be a triple sequence of $\digamma$. If $w_{nml} \to_{G_Q} w_1$ in $(\digamma, G_Q)$ and $w_{nml} \to_{G_Q} w_2$ in $(\digamma, G_Q)$, then $w_1 = w_2$.*

Proof. Let $(\digamma, G_{\mathbf{Q}})$ be a $G_{\mathbf{Q}}$ −metric space and let $\{w_{n_1m_1l_1}, w_{n_2m_2l_2}, w_{n_3m_3l_3}\} \subseteq \digamma$. By Definition 3.1 for $0 \prec q_0 \in \mathbf{Q}$, there exists N_1 and N_2 such that

$$G_{\mathbf{Q}}\left(w_1, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}\right) \prec \frac{q_0}{3} \text{ for all } n_1, n_2, m_1, m_2, l_1, l_2 > N_1$$
$$G_{\mathbf{Q}}\left(w_2, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}\right) \prec \frac{q_0}{3} \text{ for all } n_1, n_2, m_1, m_2, l_1, l_2 > N_2$$

Set $n_0 = \max\{N_1, N_2\}$. If $n, m, l \geq n_0$, then we get

$$\begin{aligned} G_{\mathbf{Q}}(w_1, w_2, w_2) &\leq G_{\mathbf{Q}}(w_1, w_{nml}, w_{nml}) + G_{\mathbf{Q}}(w_{nml}, w_2, w_2) \\ &\leq G_{\mathbf{Q}}(w_1, w_{nml}, w_{nml}) + 2G_{\mathbf{Q}}(w_2, w_{nml}, w_{nml}) \\ &< \frac{q_0}{3} + \frac{2q_0}{3} = q_0 \end{aligned}$$

Since $0 \prec q_0 \in \mathbf{Q}$ is arbitrary, $G_{\mathbf{Q}}(w_1, w_2, w_2) = 0$. Hence $w_1 = w_2$ as desired.

Proposition 3.5. *Let $(\digamma, G_Q)$ be a G_Q −metric space and $(w_{nml}) \subset \digamma$ be a triple sequence. Then (w_{nml}) is a G_Q −convergent to $w \in \digamma$ if and only if*

$$\left|G_Q\left(w, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}\right)\right| \to 0 \text{ as } n_1, n_2, m_1, m_2, l_1, l_2 \to \infty$$

Proof. Assume that (w_{nml}) is a $G_{\mathbf{Q}}$ −convergent to w. For a given $\varepsilon > 0$, let $q_0 = \frac{\varepsilon}{2} + \frac{\varepsilon}{2}i + \frac{\varepsilon}{2}j + \frac{\varepsilon}{2}k$. We have that $0_{\mathbf{Q}} \prec q_0 \in \mathbf{Q}$ and there exists $n_0 > 0$ such that $G_{\mathbf{Q}}\left(w, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}\right) \prec q_0, \forall n_1, n_2, m_1, m_2, l_1, l_2 \geq n_0$. Therefore $\left|G_{\mathbf{Q}}\left(w, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}\right)\right| < |q_0| = \varepsilon$. Consequently, $\left|G_{\mathbf{Q}}\left(w, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}\right)\right| \to 0$ as $n_1, n_2, m_1, m_2, l_1, l_2 \to \infty$. Conversely, assume that $\left|G_{\mathbf{Q}}\left(w, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}\right)\right| \to 0$ as $n_1, n_2, m_1, m_2, l_1, l_2 \to \infty$. Let $q_0 \in \mathbf{Q}$ with $0 \prec q_0$. There exists $\alpha > 0$ such that for $q_0 \in \mathbf{Q}$ such that $|q_0| < \alpha \Rightarrow q_0 \prec q_1$. Since $\left|G_{\mathbf{Q}}\left(w, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}\right)\right| \to 0$ as $n_1, n_2, m_1, m_2, l_1, l_2 \to \infty$, we have for $\alpha > 0$ there exists $n_0 \in \mathbb{N}$ for all $n_1, n_2, m_1, m_2, l_1, l_2 \geq n_0$ such that $\left|G_{\mathbf{Q}}\left(w, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}\right)\right| < \alpha$. This implies that

$$G_{\mathbf{Q}}\left(w, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}\right) \prec q_1$$

This means that (w_{nml}) is a quaternion G −convergent as required.

Corollary 3.6. *Let $(\digamma, G_Q)$ be a G_Q −metric space, $(w_{nml}) \subset \digamma$ and $w \in \digamma$. The following properties are equivalent:*

(i). (w_{nml}) *is a* G_Q *−convergent to* w.

(ii). $\left|\boldsymbol{G_Q}(w_{nml}, w_{nml}, w)\right| \to 0$ *as* $n, m, l \to \infty$.

(iii). $\left|G_Q(w_{nml}, w, w)\right| \to 0$ *as* $n, m, l \to \infty$.

(iv). $\left|G_{\boldsymbol{Q}}\left(w_{n_1m_1l_1}, w_{n_2m_2l_2}, w\right)\right| \to 0$ *as* $n_1, n_2, m_1, m_2, l_1, l_2 \to \infty$.

Proposition 3.7. *Let* $(\digamma, G_{\boldsymbol{Q}})$ *be a* $G_{\boldsymbol{Q}}$ *−metric space and* $(w_{nml}) \subset \digamma$ *be a triple sequence. Then* (w_{nml}) *is a* $G_{\boldsymbol{Q}}$ *−Cauchy sequence if and only if*

$$\left|G_{\boldsymbol{Q}}\left(w_{n_0m_0l_0}, w_{n_1m_1l_1}, w_{n_2m_2l_2}\right)\right| \to 0 \text{ as } n_1, m_1, n_2, m_2, l_1, l_2 \to \infty.$$

Proof. Suppose that (w_{nml}) is $G_{\mathbf{Q}}$ −Cauchy sequence. For a given $\varepsilon > 0$, let $q_0 = \frac{\varepsilon}{2} + \frac{\varepsilon}{2}i + \frac{\varepsilon}{2}j + \frac{\varepsilon}{2}k$. We have that $0_{\mathbf{Q}} \prec q_0 \in \mathbf{Q}$ and there exists $N > 0$ such that

$$G_{\mathbf{Q}}\left(w_{n_0m_0l_0}, w_{n_1m_1l_1}, w_{n_2m_2l_2}\right) \prec q_0, \forall n_1, n_2, m_1, m_2, l_1, l_2 \geq N$$

Therefore $\left|G_{\mathbf{Q}}\left(w_{n_0m_0l_0}, w_{n_1m_1l_1}, w_{n_2m_2l_2}\right)\right| < |q_0| = \varepsilon$. Consequently,

$$\left|G_{\mathbf{Q}}\left(w_{n_0m_0l_0}, w_{n_1m_1l_1}, w_{n_2m_2l_2}\right)\right| \to 0$$

as $n_1, n_2, m_1, m_2, l_1, l_2 \to \infty$

Conversely, suppose that

$$\left|G_{\mathbf{Q}}\left(w_{n_0m_0l_0}, w_{n_1m_1l_1}, w_{n_2m_2l_2}\right)\right| \to 0$$

as $n_1, n_2, m_1, m_2, l_1, l_2 \to \infty$. Let $q_0 \in \mathbf{Q}$ with $0 \prec q_0$. There exists $\alpha > 0$ such that for $q \in \mathbf{Q}$ such that $|q| < \alpha \Rightarrow q \prec q_0$. Since $\left|G_{\mathbf{Q}}\left(w_{n_0m_0l_0}, w_{n_1m_1l_1}, w_{n_2m_2l_2}\right)\right| \to 0$ as $n_1, n_2, m_1, m_2, l_1, l_2 \to \infty$, we have for $\alpha > 0$ there exists $N \in \mathbb{N}$ for all $n, m, l \geq N$ such that

$$\left|G_{\mathbf{Q}}\left(w_{n_0m_0l_0}, w_{n_1m_1l_1}, w_{n_2m_2l_2}\right)\right| < \alpha$$

Implying that

$$G_{\mathbf{Q}}\left(w_{n_0m_0l_0}, w_{n_1m_1l_1}, w_{n_2m_2l_2}\right) \prec q_0$$

This means that (w_{nml}) is $G_{\mathbf{Q}}$ −Cauchy sequence as required.

4. Statistical convergence of G_Q −metric space

In this section, we introduce statistical convergence in $G_{\mathbf{Q}}$ −metric spase and give several key characteristics.

Definition 4.1. *Let* $(\digamma, G_Q)$ *be a* G_Q *−metric space and* $(w_{nml}) \subset \digamma$ *be a triple sequence.* $\{w_{nml}\}$ *statistically converges to* w *if for every* $q_0 \prec \boldsymbol{Q}$ *with* $0 \prec q_0$ *such that*

$$\lim_{n,m,l\to\infty} \frac{2}{(nml)^2}\Big| \{(n_1,n_2,n_3),(m_1,m_2,m_3),(l_1,l_2,l_3) \in \mathbb{N}^3 \times \mathbb{N}^3, n_1,n_2,n_3 \leq n$$
$$m_1,m_2,m_3 \leq m, l_1,l_2,l_3 \leq l: \left|G_{\boldsymbol{Q}}\left(w, w_{n_1m_1l_1}, w_{n_2m_2l_2}\right)\right| \geq |q_0|\} \mid = 0,$$

and denoted by $G_{\boldsymbol{Q}}(st) - lim_{n\to\infty}\{w_{n_1m_1l_1}, w_{n_2m_2l_2}\} = w$ *or* $\{w_{nml}\} \overset{G_{H(st)}}{\to} w$.

Definition 4.2. *Let* $(\digamma, G_Q)$ *be a* G_Q *−metric space.* $\{w_{nml}\}$ *is said to be statistically* G_Q *−Cauchy if for every* $q_0 \in \boldsymbol{Q}$ *with* $0 \prec q_0$*, there exists* $n_0, m_0, l_0 \prec \boldsymbol{Q}$ *such that*

$$\lim_{n,m,l\to\infty} \frac{2}{(nml)^2}\Big| \{(n_1,n_2,n_3),(m_1,m_2,m_3),(l_1,l_2,l_3) \in \mathbb{N}^3 \times \mathbb{N}^3, n_1,n_2,n_3 \leq n$$
$$m_1,m_2,m_3 \leq m, l_1,l_2,l_3 \leq l: \left|G_{\boldsymbol{Q}}\left(w_{n_0m_0l_0}, w_{n_1m_1l_1}, w_{n_2m_2l_2}\right)\right| \geq |q_0|\} \mid = 0.$$

Definition 4.3. *Let* $(\digamma, G_Q)$ *be a* G_Q *−metric space.* $(\digamma, G_Q)$ *is called a complete quaternion valued* G *−metric space if every statistically* G_Q *−Cauchy sequence in*$(\digamma, G_Q)$ *is* $G_Q(st)$ *−convergence in* $(\digamma, G_Q)$.

Definition 4.4. *Let* $(\digamma, G_Q)$ *be a* G_Q *−metric space.* $\{w_{nml}\}$ *is bounded if there exists a positive number* M *such that* $\left|\{w_{n_1m_1l_1}, w_{n_2m_2l_2}, w_{n_3m_3l_3}\}\right| \leq M$ *for all* $\{(n_1,n_2,n_3),(m_1,m_2,m_3),(l_1,l_2,l_3)\}$.

We will denote the set of all bounded sequences by ℓ_∞^G.

Theorem 4.5. *If a triple sequence* $\{w_{nml}\}$ *is statistically convergent in* $(\digamma, G_Q)$ *then* $G_Q(st) - lim(w_{nml})$ *is unique.*

Proof. Suppose that $\{w_{nml}\}$ statistically converges in $(\digamma, G_{\mathbf{Q}})$. Let $G_{\mathbf{Q}}(st) - \lim(w_{nml}) = L_1$ and $G_{\mathbf{Q}}(st) - \lim(w_{nml}) = L_2$. Given $\varepsilon > 0$ and $0 \prec q_0 \in \mathbf{Q}$, let

$$q_0 = \frac{\varepsilon}{4p} + i\frac{\varepsilon}{4p} + j\frac{\varepsilon}{4p} + k\frac{\varepsilon}{4p}$$

Define the following sets as:

$$S_1(\varepsilon)=\{(n_1,n_2,n_3),(m_1,m_2,m_3),(l_1,l_2,l_3)\in\mathbb{N}^3\times\mathbb{N}^3, n_1,n_2,n_3\leq n$$
$$m_1,m_2,m_3\leq m, l_1,l_2,l_3\leq l: \left|G_{\mathbf{Q}}\left(L_1,w_{n_1m_1l_1},w_{n_2m_2l_2}\right)\right|\geq|q_0|=\frac{\varepsilon}{2p}\right\}$$
$$S_2(\varepsilon)=\{(n_1,n_2,n_3),(m_1,m_2,m_3),(l_1,l_2,l_3)\in\mathbb{N}^3\times\mathbb{N}^3, n_1,n_2,n_3\leq n$$
$$m_1,m_2,m_3\leq m, l_1,l_2,l_3\leq l: \left|G_{\mathbf{Q}}\left(L_2,w_{n_1m_1l_1},w_{n_2m_2l_2}\right)\right|\geq|q_0|=\frac{\varepsilon}{2p}\right\}$$

Since $G_{\mathbf{Q}}(st)-\text{Iim}(w_{nml})=L_1$, we have $\delta(S_1(\varepsilon))=0$. Similarly $G_{\mathbf{Q}}(st)-\lim(w_{nml})=L_2$, implies $\delta(S_2(\varepsilon))=0$.

Let $S(\varepsilon)=S_1(\varepsilon)\cup S_2(\varepsilon)$. Then $\delta(S(\varepsilon))=0$ and we have $S^c(\varepsilon)$ is non-empty and $\delta(S^c(\varepsilon))=1$. Suppose that $\{n_1,n_2,n_3\},\{m_1,m_2,m_3\},\{l_1,l_2,l_3\}\in S^c(\varepsilon)$. Then we have

$$\begin{aligned}
\left|G_{\mathbf{Q}}(L_1,L_2,L_2)\right| &\leq \left|G_{\mathbf{Q}}(L_1,w_{nml},w_{nml})\right|+\left|G_{\mathbf{Q}}(w_{nml},L_2,L_2)\right|\\
&\leq \left|G_{\mathbf{Q}}(L_1,w_{nml},w_{nml})\right|+2\left|G_{\mathbf{Q}}(L_2,w_{nml},w_{nml})\right|\\
&\leq \left|G_{\mathbf{Q}}\left(L_1,w_{n_1m_1l_1},w_{n_2m_2l_2}\right)\right|+2\left|G_{\mathbf{Q}}\left(L_2,w_{n_1m_1l_1},w_{n_2m_2l_2}\right)\right|\\
&\leq 2\left|G_{\mathbf{Q}}\left(L_1,w_{n_1m_1l_1},w_{n_2m_2l_2}\right)\right|+2\left|G_{\mathbf{Q}}\left(L_2,w_{n_1m_1l_1},w_{n_2m_2l_2}\right)\right|\\
&\leq 2\left(\frac{\varepsilon}{2p}+\frac{\varepsilon}{2p}\right)=\varepsilon
\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, we get $G_{\mathbf{Q}}(L_1,L_2,L_2)=0$, therefore $L_1=L_2$.

Theorem 4.6. *If $G_Q-limw_{nml}=w$ then $G_Q(st)-limw_{nml}=w$ but converse need not be true in general.*

Proof. Let $G_{\mathbf{Q}}-\lim w_{nml}=w$. Then for all $0\prec q_0\in\mathbf{Q}$ there exists $N\in\mathbb{N}$ such that.

$$n_1,n_2,m_1,m_2,l_1,l_2\geq N\Rightarrow G_{\mathbf{Q}}\left(w,w_{n_1m_1l_1},w_{n_2m_2l_2}\right)\prec q_0$$

The set

$$A(\varepsilon)=\{(n_1,n_2,n_3),(m_1,m_2,m_3),(l_1,l_2,l_3)\in\mathbb{N}^3\times\mathbb{N}^3, n_1,n_2,n_3\leq n$$
$$m_1,m_2,m_3\leq m, l_1,l_2,l_3\leq l: \left|G_{\mathbf{Q}}\left(w,w_{n_1m_1l_1},w_{n_2m_2l_2}\right)\right|\geq|q_0|=\varepsilon\}$$
$$\subset\{(1,1,1),(2,2,2),\ldots\}^2$$

where $q=\frac{\varepsilon}{2}+i\frac{\varepsilon}{2}+j\frac{\varepsilon}{2}+k\frac{\varepsilon}{2}, \delta(A(\varepsilon))=0$. Hence $G_{\mathbf{Q}}(st)-\lim w_{nml}=w$.

The following example shows that the converse need not be true.

Example 4.7. Take the quaternion valued G −metric space in Example 2.3. Let w_{nml} be a triple sequence defined as

$$w_{nml}=\begin{cases} nml, & \text{if } n,m,l \text{ is a cube,} \\ 1, & \text{otherwise.}\end{cases}$$

It is easy to see that $G_{\mathbf{Q}}(st)-\lim w_{nml}=1$, since the cardinality of the set

$\{(n,m,l)\in\mathbb{N}^3\times\mathbb{N}^3, m\le\alpha, n\le\beta, l\le\gamma(\alpha,\beta,\gamma\in\mathbb{N}):|G_{\mathbf{Q}}(1,w_{nml})|>|q|=\varepsilon\}$, where

$$q=\frac{\varepsilon}{2}+i\frac{\varepsilon}{2}+j\frac{\varepsilon}{2}+k\frac{\varepsilon}{2}\le\sqrt{nml}$$

for every $\varepsilon>0$. But $\{w_{nml}\}$ is neither convergent nor bounded.

Theorem 4.8. *Let $(\digamma,G_{\boldsymbol{Q}})$ be a $G_{\boldsymbol{Q}}$ −metric space Then a triple sequence $\{w_{nml}\}$ of points in $(\digamma,G_{\boldsymbol{Q}})$ is statistically $G_{\boldsymbol{Q}}$ −convergent if and only if it is statistically $G_{\boldsymbol{Q}}$ −Cauchy.*

Proof. Suppose that $G_{\mathbf{Q}}(st)-\lim w_{nml}=w$. Then, we get $\delta(A(\varepsilon))=0$, where

$$A(\varepsilon)=\{(n_1,n_2,n_3),(m_1,m_2,m_3),(l_1,l_2,l_3)\in\mathbb{N}^3\times\mathbb{N}^3, n_1,n_2,n_3\le n \\ m_1,m_2,m_3\le m, l_1,l_2,l_3\le l: |G_{\mathbf{Q}}(w,w_{n_1m_1l_1},w_{n_2m_2l_2})|\ge|q_0|=\frac{\varepsilon}{2}\}$$

where $q_0=\frac{\varepsilon}{4}+i\frac{\varepsilon}{4}+j\frac{\varepsilon}{4}+k\frac{\varepsilon}{4}$.

This implies that

$$\delta(A^c(\varepsilon))=\{(n_1,n_2,n_3),(m_1,m_2,m_3),(l_1,l_2,l_3)\in\mathbb{N}^3\times\mathbb{N}^3, n_1,n_2,n_3\le n \\ m_1,m_2,m_3\le m, l_1,l_2,l_3\le l: |G_{\mathbf{Q}}(w,w_{n_1m_1l_1},w_{n_2m_2l_2})|<|q_0|=\frac{\varepsilon}{2}\}=1$$

Let $(j_1,j_2,j_3),(k_1,k_2,k_3),(v_1,v_2,v_3)\in A^n(\varepsilon)$. Then

$$|G_{\mathbf{Q}}(w,w_{j_1k_1v_1},w_{j_2k_2v_2})|<|q_0|=\frac{\varepsilon}{2}$$

Let

$$B(\varepsilon)=\{(n_1,n_2,n_3),(m_1,m_2,m_3),(l_1,l_2,l_3)\in\mathbb{N}^3\times\mathbb{N}^3, n_1,n_2,n_3\le n, m_1,m_2,m_3\le m \\ l_1,l_2,l_3\le l: |G_{\mathbf{Q}}(w_{j_1k_1v_1},w_{j_2k_2v_2},w_{j_3k_3v_3},w_{n_1m_1l_1},w_{n_2m_2l_2},w_{n_3m_3l_3})|\ge\varepsilon\},$$

we need to show that $B(\varepsilon) \subset A(\varepsilon)$. Let $n, m \in B(\varepsilon)$. Hence $\left|G_{\mathbf{Q}}\left(w, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}\right)\right| \geq \frac{\varepsilon}{2}$. Otherwise if $\left|G_{\mathbf{Q}}\left(w, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}\right)\right| \leq \varepsilon$ then

$$\begin{aligned}
\varepsilon &\leq \left|G_{\mathbf{Q}}\left(w_{j_1 k_1 v_1}, w_{j_2 k_2 v_2}, w_{j_3 k_3 v_3}, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}, w_{n_3 m_3 l_3}\right)\right| \\
&\leq G_{\mathbf{Q}}\left(w, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}\right) + G_{\mathbf{Q}}\left(w, w_{j_1 k_1 v_1}, w_{j_2 k_2 v_2}\right) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

which is not possible. Hence $B(\varepsilon) \subset A(\varepsilon)$, which implies that the triple sequence $\{w_{nml}\}$ is statistically $G_{\mathbf{Q}}$ −Cauchy.

Conversely, suppose that $\{w_{nml}\}$ is $G_{\mathbf{Q}}(st)$ −Cauchy but not $G_{\mathbf{Q}}(st)$ −convergent. Then, there exists $(j_1, j_2, j_3), (k_1, k_2, k_3), (v_1, v_2, v_3) \in \mathbb{N}^3 \times \mathbb{N}^3$ such that $\delta(G(\varepsilon)) = 0$ where

$$\begin{gathered}
G(\varepsilon) = \{(n_1, n_2, n_3), (m_1, m_2, m_3), (l_1, l_2, l_3) \in \mathbb{N}^3 \times \mathbb{N}^3, n_1, n_2, n_3 \leq n, m_1, m_2, m_3 \leq m \\
l_1, l_2, l_3 \leq l : \left|G_{\mathbf{Q}}\left(w_{j_1 k_1 v_1}, w_{j_2 k_2 v_2}, w_{j_3 k_3 v_3}, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}, w_{n_3 m_3 l_3}\right)\right| \geq \varepsilon\}
\end{gathered}$$

and $\delta(D(\varepsilon)) = 0$, where

$$\begin{gathered}
G(\varepsilon) = \{(n_1, n_2, n_3), (m_1, m_2, m_3), (l_1, l_2, l_3) \in \mathbb{N}^3 \times \mathbb{N}^3, n_1, n_2, n_3 \leq n \\
m_1, m_2, m_3 \leq m, l_1, l_2, l_3 \leq l : \left|G_{\mathbf{Q}}\left(w, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}\right)\right| < \frac{\varepsilon}{2}\},
\end{gathered}$$

that is $\delta(D^c(\varepsilon)) = 1$. Since

$$\begin{aligned}
&\left|G_{\mathbf{Q}}\left(w_{j_1 k_1 v_1}, w_{j_2 k_2 v_2}, w_{j_3 k_3 v_3}, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}, w_{n_3 m_3 l_3}\right)\right| \\
&\leq 2\left|G_{\mathbf{Q}}\left(w, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}\right)\right| \leq \varepsilon,
\end{aligned}$$

we have $\left|G_{\mathbf{Q}}\left(w, w_{n_1 m_1 l_1}, w_{n_2 m_2 l_2}\right)\right| < \frac{\varepsilon}{2}$. Therefore $\delta(G^c(\varepsilon)) = 0$ that is $\delta(G(\varepsilon)) = 1$, which leads the contradiction, since $\{w_{nml}\}$ was $G_{\mathbf{Q}}(st)$ −Cauchy. Hence $\{w_{nml}\}$ is $G_{\mathbf{Q}}(st)$ −convergent.

Now, we establish the relation between strong summability and $G_{\mathbf{Q}}$ −statistical convergence in $G_{\mathbf{Q}}$ −metric space

Definition 4.9. *A triple sequence* $\{w_{nml}\}$ *is said to be strongly* p −*Cesaro summable* $(0 < p < \infty)$ *to limit* w *in* $(\digamma, G_Q)$ *if*

$$\lim_{n,m,l\to\infty} \frac{2}{(nml)^2} \sum_{n_1,n_2,m_1,m_2,l_1,l_2=1}^{n,m,l} \left(G_{\boldsymbol{Q}}\left(w_{n_1m_1l_1}, w_{n_2m_2l_2}, w\right)^p \right) = 0$$

and write it as $w_{nml} \to w\left[C_1, G_{\boldsymbol{Q}}\right]_p$. *In this case* w *is the* $\left[C_1, G_{\boldsymbol{Q}}\right]_p$ *−limit of* $\{w_{nml}\}$.

Theorem 4.10. (a) *If* $0 < p < \infty$ *and* $\{w_{nml}\} \to w\left[C_1, G_{\boldsymbol{Q}}\right]_p$, *then* $\{w_{nml}\}$ *is statistically* $G_{\boldsymbol{Q}}$ *−convergent to* w *in* $\left(\digamma, G_{\boldsymbol{Q}}\right)$.

(b) *If* $\{w_{nml}\}$ *is bounded and* $G_{\boldsymbol{Q}}$ *−statistically convergent to* w *in* $\left(\digamma, G_{\boldsymbol{Q}}\right)$, *then* $w_{nml} \to w\left[C_1, G_{\boldsymbol{Q}}\right]_p$.

Proof. (a) Let

$$S_\varepsilon(p) = \{(n_1,n_2,n_3),(m_1,m_2,m_3),(l_1,l_2,l_3) \in \mathbb{N}^3 \times \mathbb{N}^3, n_1,n_2,n_3 \le n \\ m_1,m_2,m_3 \le m, l_1,l_2,l_3 \le l: \left|G_{\mathbf{Q}}\left(w_{n_1m_1l_1}, w_{n_2m_2l_2}, w\right)\right|^p \ge \varepsilon\}$$

Now since $w_{nml} \to w\left[C_1, G_{\mathbf{Q}}\right]_p$ then

$$\begin{aligned}
0 \longleftarrow & \frac{2}{(nml)^2} \sum_{n_1,n_2,m_1,m_2,l_1,l_2=1}^{n,m,l} \left(G_{\mathbf{Q}}\left(w_{n_1m_1l_1}, w_{n_2m_2l_2}, w\right)^p \right) \\
= & \frac{2}{(nm)^2} \left\{ \sum_{n_1,n_2,m_1,m_2,l_1,l_2=1, n_i,m_i,l_i \notin S_\varepsilon(p)}^{n,m,l} \left(G_{\mathbf{Q}}\left(w_{n_1m_1l_1}, w_{n_2m_2l_2}, w\right)^p \right) \right. \\
& \left. + \sum_{n_1,n_2,m_1,m_2,l_1,l_2=1, n_i,m_i,l_i \in S_\varepsilon(p)}^{n,m,l} \left(G_{\mathbf{Q}}\left(w_{n_1m_1l_1}, w_{n_2m_2l_2}, w\right)^p \right) \right\} \\
\ge & \frac{2}{(nml)^2} |S_\varepsilon(p)| \varepsilon^p, \text{ as } n,m,l \to \infty
\end{aligned}$$

That is, $\lim_{n,m,l\to\infty} \frac{2}{(nml)^2} |S_\varepsilon(p)| = 0$ and $\delta(S_\varepsilon(p)) = 0$.

(b) Suppose that $\{w_{nml}\}$ is bounded and $G_{\mathbf{Q}}$ −statistically convergent to w in $\left(\digamma, G_{\mathbf{Q}}\right)$. Then for $\varepsilon > 0$, we have $\delta(S_\varepsilon(p)) = 0$. Also, there exists $M > 0$ such that $\left|G_{\mathbf{Q}}\left(w_{n_1m_1l_1}, w_{n_2m_2l_2}, w\right)\right|^P \le M$. We have

$$\frac{2}{(nml)^2}\sum_{n_1,n_2,m_1,m_2=1}^{n,m}\left(G_{\mathbf{Q}}\left(w_{n_1m_1l_1},w_{n_2m_2l_2},w\right)^p\right)$$
$$=\frac{2}{(nml)^2}\sum_{n_1,n_2,m_1,m_2,l_1,l_2=1,n_i,m_i,l_i\notin S_\varepsilon(p)}^{n,m,l}\left(G_{\mathbf{Q}}\left(w_{n_1m_1l_1},w_{n_2m_2l_2},w\right)^p\right)$$
$$+\frac{2}{(nml)^2}\sum_{n_1,n_2,m_1,m_2,l_1,l_2=1,n_i,m_i,l_i\in S_\varepsilon(p)}^{n,m,l}\left(G_{\mathbf{Q}}\left(w_{n_1m_1l_1},w_{n_2m_2l_2},w\right)^p\right)$$
$$=S_1(\varepsilon)+S_2(\varepsilon)$$

where

$$T_1(\varepsilon)=\frac{2}{(nml)^2}\sum_{n_1,n_2,m_1,m_2,l_1,l_2=1,n_i,m_i,l_i\notin S_\varepsilon(p)}^{n,m,l}\left(G_{\mathbf{Q}}\left(w_{n_1m_1l_1},w_{n_2m_2l_2},w\right)^p\right)$$

and

$$T_2(\varepsilon)=\frac{2}{(nml)^2}\sum_{n_1,n_2,m_1,m_2,l_1,l_2=1,n_i,m_i,l_i\in S_\varepsilon(p)}^{n,m,l}\left(G_{\mathbf{Q}}\left(w_{n_1m_1l_1},w_{n_2m_2l_2},w\right)^p\right)$$

Now if $(n_1,n_2,n_3),(m_1,m_2,m_3),(l_1,l_2,l_3)\notin S_\varepsilon(p)$ then $T_1(\varepsilon)<\varepsilon^p$. For (n_1,n_2,n_3), $(m_1,m_2,m_3),(l_1,l_2,l_3)$, we have

$$T_2(\varepsilon)\leq \sup\left|G_{\mathbf{Q}}\left(w_{n_1m_1l_1},w_{n_2m_2l_2},w\right)\right|\left(\frac{2|S_\varepsilon(p)|}{(nml)^2}\right)\leq M\frac{2|S_\varepsilon(p)|}{(nml)^2}\to 0$$

as $n,m,l\to\infty$, since $\delta(S(\varepsilon))=0$. Hence $w_{nml}\to w\left[C_1,G_{\mathbf{Q}}\right]_p$.

5. Conclusion

We have presented and discussed convergence for triple sequences in $G_{\mathbf{Q}}$ −metric space in this work, looking at a number of fundamental characteristics. In addition, we have carried out an extensive examination of statistical convergence within this framework with the goal of offering a thorough comprehension of sequence behavior within this particular metric environment. We have concluded our study by examining the connection between strong summability and statistical convergence in $G_{\mathbf{Q}}$ −metric spaces. These results highlight the close relationships between summability and statistical convergence, providing important new information on the nature of convergence in G_{Q} −metric spaces.

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