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Research Article

Discussion of $(2+1)$ dimensional mixed integral equation with singular kernel

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Abstract: In this paper, we discuss the uniqueness and existence of solution for a quadratic mixed integral equation (QMIE) on $2+1$ dimensional in $L_2([0,a] \times [0,b] \times C[0,T])$.Where, $(T<1)$ space using fixed point theorem. Further we demonstrate convergence of the given existence result. Also we discuss the theoretical part of the Chybeshev polynomial method extending in $2+1$ dimensional format and finally discuss the error analysis of the method.

Keywords: Quadratic mixed integral equation (QMIE), Volterra Fredholm Integral equation, Banach FPT, Singular kernel; Chebyshev Polynomial, FPT (Fixed point theorem).

MSC Subject Classification: 47H10, 45G05,45G10, 45E05, 65R20.

1. Introduction

The Volterra-Fredholm integral equations comes from problems of parabolic boundary value, mathematical modeling based on the spatio-temporal development of an epidemic, from a variety of physical and biological models. Applications of Volterra-Fredholm integral equations typically occur in fields related to physics, fluid dynamics, electrodynamics and biology. Detailed of Mixed Volterra Fredholm integral equation will be found in **(Wazwaz, 2011)**. Numerous publications have recently been published that focus on understanding these equations and their properties. Many researchers have shown immense interests on this issue and many generalizations of the same have been given by a lot of researchers.Some researchers investigated of mixed integrral equation by taking properties of kernal. Some of these research papers which will help us in this discussion are **(Albugami et al., 2024; Alharbi, 2024; Mahdy et al. 2024; Noeiaghdam, 2021)**. Lately, a few researchers started investigating that how time function affect the solution of integral equation **(Abusalim, 2023;**

Al Hazmi, 2023; Al Hazmi et al., 2023; Jan, 2002a; Jan, 2002b; Mahdy et al., 2024; Mahdy et al., 2023; Matoog et al., 2023).

In this paper, we consider QMIE with singular kernel in $2 + 1$ dimension. Here, the position kernel can take form as Hilbert kernel, Carleman kernel form, Cauchy kernel form or logarithm form.

Consider in the space $L_2([0, a] \times [0, b]) \times C[0, T], (T < 1)$ the $(2 + 1)$ dimensional with singular kernel

$$\xi(u_x, u_y)\varphi(u_x, u_y; t) = \mathcal{F}(u_x, u_y; t) + \Gamma(u_x, u_y)\int_0^t \int_0^a \int_0^b g(t, \tau)\kappa(\omega(u) - \omega(v))\varphi(v_x, v_y; \tau)dv_x dv_y d\tau, \tag{1}$$

where $u = (u_x, u_y)$ and $v = (v_x, v_y)$ are 2 dimensional spatial variables. In equation (1) $\mathcal{F}(u_x, u_y; t)$ is a known continuous function that explain the type of QMIE indicated by the symbol $\xi(u_x, u_y)$.$\Gamma(u_x, u_y)$ is a function which has various physical interpretations in the field of genetic engineering and many other. $g(t, \tau)$ is a time kernel and $\kappa(\omega(u) - \omega(v))$ is a position kernel. Here we take the singular position kernel. And the function $\varphi(u_x, u_y; t)$ is unknown function.

2. Existence of unique solution

For verifying existence and uniqueness of the equation (1), we will use Banach fixed point theorem. If we represent our equation (1) into an integral operator, then we get

$$\bar{S}\varphi(u_x, u_y; t) = \frac{\mathcal{F}(u_x, u_y; t)}{\xi(u_x, u_y)} + \frac{\Gamma(u_x, u_y)}{\xi(u_x, u_y)} S\varphi(u_x, u_y; t), (\xi(u_x, u_y) \neq 0, \Gamma(u_x, u_y) \neq 0), \tag{2}$$

$$S\varphi(u_x, u_y; t) = \int_0^t \int_0^a \int_0^b g(t, \tau)\kappa(\omega(u) - \omega(v))\varphi(v_x, v_y; \tau)dv_x dv_y d\tau. \tag{3}$$

Now, assume the following conditions:

(i). The position kernel $\kappa(\omega(u) - \omega(v))$ fulfilled the condition of discontinuity

$$\left(\int_0^a \int_0^b \int_0^a \int_0^b \left|\kappa\left(\omega(u) - \omega(v)\right)\right|^2 du_x du_y dv_x dv_y\right)^{1/2} = \mathcal{C}.$$

(ii). The time kernel fulfill $|g(t, \tau)| \leq M, \forall t \in [0, T], t < 1$, where $g(t, \tau) \in C([0, T] \times [0, T])$; while $\left|\xi(u_x, u_y)\right| \leq \bar{w}, \left|\Gamma(u_x, u_y)\right| \leq \bar{\Gamma}$.

(iii). The norm of function $F(u_x, u_y; t)$ is in the space $L_2([0,a] \times [0,b]) \times C[0,T], (T < 1)$ and fulfills

$$\left\|F(u_x, u_y; t)\right\|_{L_2([0,a]\times[0,b])\times C[0,T]} = \max_{0\le t\le T<1} \int_0^t \left(\int_0^a \int_0^b \left(F(u_x, u_y; \tau)\right)^2 du_x du_y\right)^{1/2} d\tau = \mathcal{G}.$$

(iv). The unknown function $\varphi(u_x, u_y; t)$, the norm is $\left\|\varphi(u_x, u_y; t)\right\| = \mathcal{Q}$.

In the conditions (i)-(iv), $\mathcal{C}, \mathcal{M}, \overline{w}, \overline{\Gamma}, \mathcal{G}$ and $\mathcal{Q}$ are positive constants.

Theorem 2.1. *By the above assumptions, the QMIE* (1.1) *has a unique solution if*

$$\overline{\Gamma}\mathcal{M}\mathcal{C}T < \overline{w}.$$

To show the existence and uniqueness of the solution of equation (1) we must determine the following two lemmas:

Lemma 2.2. *Under the assumptions* (i)-(iv), *the operator* $\overline{S}$, *is a itself map on the space* $L_2([0,a] \times [0,b]) \times C[0,T], T < 1$.

Proof. Using the assumptions (i) and (iii), we have

$$\begin{aligned}
||\overline{S}\varphi(u_x, u_y; t)|| &= ||\frac{\mathcal{F}(u_x, u_y; t)}{\xi(u_x, u_y)} + \frac{\Gamma(u_x, u_y)}{\xi(u_x, u_y)} S\varphi(u_x, u_y; t)|| \\
&\le \frac{\mathcal{G}}{|\xi(u_x, u_y)|} \\
&+ \frac{|\Gamma(u_x, u_y)|}{|\xi(u_x, u_y)|} || \int_0^t \int_0^a \int_0^b g(t,\tau)\kappa(\omega(u) - \omega(v))\varphi(v_x, v_y; \tau) dv_x dv_y d\tau|| \\
&\le \frac{\mathcal{G}}{\overline{w}} + \frac{\overline{\Gamma}}{\overline{w}}\mathcal{M}[|| \int_0^a \int_0^b \int_0^a \int_0^b |\kappa(\omega(u) - \omega(v))|^2 du_x du_y dv_x dv_y||]^{1/2} \qquad (4) \\
&\times ||max_{0\le t\le T} \int_0^t \{\int_0^a \int_0^b \{\varphi(v_x, v_y; \tau)\}^2 dv_x dv_y\}^{1/2} d\tau \\
&\le \frac{\mathcal{G}}{\overline{w}} + \frac{\overline{\Gamma}}{\overline{w}}\mathcal{M}\mathcal{C}T||\varphi(u_x, u_y; t)|| \\
&\le \frac{\mathcal{G}}{\overline{w}} + \sigma \left\|\varphi(u_x, u_y; t)\right\|, (\sigma = \frac{\overline{\Gamma}}{\overline{w}}\mathcal{M}\mathcal{C}T)
\end{aligned}$$

The inequality (4) shows that, the operator $\overline{S}$ maps the ball into itself

$$\varrho = \frac{\mathcal{G}}{[\overline{w} - \overline{\Gamma}\mathcal{MC}T]}.$$

Since $\varrho > 0, \mathcal{G} > 0$, therefore we get $\sigma < 1$. Then, we have

$$||S\varphi(u_x, u_y; t)|| = ||S||||\varphi(u_x, u_y; t)|| \leq \sigma||\varphi(u_x, u_y; t)||.$$

Lemma 2.3. *If we consider the assumptions* (i), (ii) *and* (iv) *then the integral operator* $\overline{S}$ *is continuous in the space* $L_2([0, a] \times [0, b]) \times C[0, T]$.

Proof. Consider two functions $\varphi_1(u_x, u_y; t)$ and $\varphi_2(u_x, u_y; t)$ in the space $L_2([0, a] \times [0, b]) \times C[0, T]$, then we have

$$||\overline{S}\varphi_1(u_x, u_y; t) - \overline{S}\varphi_2(u_x, u_y; t)||$$
$$\leq \frac{|\Gamma(u_x, u_y)|}{|\xi(u_x, u_y)|} || \int_0^t \int_0^a \int_0^b |g(t, \tau)||\kappa(\omega(u) - \omega(v))||\varphi_1(v_x, v_y; \tau) - \varphi_2(v_x, v_y; \tau)| dv_x dv_y d\tau||.$$

Taking the assumptions (ii) and (iv), we have

$$||\overline{S}\varphi_1(u_x, u_y; t) - \overline{S}\varphi_2(u_x, u_y; t)||$$
$$\leq \frac{\overline{\Gamma}}{\overline{w}} M || \int_0^t \int_0^a \int_0^b |\kappa(\omega(u) - \omega(v))||\varphi_1(v_x, v_y; \tau) - \varphi_2(v_x, v_y; \tau)| dv_x dv_y d\tau||.$$

Applying Hölder's inequality to the integral term, and then using (i), we finally obtain

$$\left\|\bar{S}\varphi_1(u_x, u_y; t) - \bar{S}\varphi_2(u_x, u_y; t)\right\| \leq \sigma||\varphi_1(v_x, v_y; \tau) - \varphi_2(v_x, v_y; \tau)||.$$

With this inequality, we can see that the operator $\overline{S}$ is continuous in $L_2([0, a] \times [0, b]) \times C[0, T]$ and then $\overline{S}$ is a contraction operator under $\sigma < 1$. Thus, by Banach's fixed point theorem, the map $\overline{S}$ itself is a unique solution of the equation (2).

3. The convergence of solution

In order to clarify why the solution is converged to $\varphi(u_x, u_y, y)$ we construct the family of solution sequence as

$$\varphi(u_x, u_y; t) = \{\varphi_0(u_x, u_y; t), \varphi_1(u_x, u_y; t), \ldots, \varphi_{n-1}(u_x, u_y; t), \varphi_n(u_x, u_y; t), \ldots\}$$

which simply can denote as $\varphi(u_x, u_y; t) = \{\varphi_n(u_x, u_y; t)\}_{n=0}^{\infty}$. Then, we pick two different functions $\varphi_{n-1}(u_x, u_y; t), \varphi_n(u_x, u_y; t)$, to construct the sequence of integral equations as below:

$$\xi(u_x, u_y)\varphi_n(u_x, u_y; t) = \mathcal{F}(u_x, u_y; t) + \Gamma(u_x, u_y)\int_0^t \int_0^a \int_0^b g(t,\tau)\kappa(\omega(u) - \omega(v))\varphi_{n-1}(v_x, v_y; \tau)dv_x dv_y d\tau$$

and

$$\xi(u_x, u_y)\varphi_{n-1}(u_x, u_y; t) = \mathcal{F}(u_x, u_y; t) + \Gamma(u_x, u_y)\int_0^t \int_0^a \int_0^b g(t,\tau)\kappa(\omega(u) - \omega(v))\varphi_{n-2}(v_x, v_y; \tau)dv_x dv_y d\tau.$$

Considering the above mentioned integral we deduce that

$$\xi(u_x, u_y)\Psi_n(u_x, u_y; t) = \Gamma(u_x, u_y)\int_0^t \int_0^a \int_0^b g(t,\tau)\kappa(\omega(u) - \omega(v))\Psi_{n-1}(v_x, v_y; \tau)dv_x dv_y d\tau,$$

where

$$\Psi_n(u_x, u_y; t) = \varphi_n(u_x, u_y; t) - \varphi_{n-1}(u_x, u_y; t); \Psi_0(u_x, u_y; t) = \frac{\mathcal{F}(u_x, u_y; t)}{\xi(u_x, u_y)}, \xi(u_x, u_y \neq 0). \quad (5)$$

Next we deduce easily

$$\varphi_n(u_x, u_y; t) = \sum_{i=0}^{n} \Psi_i(u_x, u_y; t); n = 1,2,3, \dots \quad (6)$$

Using (5) we have

$$||\Psi_n(u_x, u_y; t)|| \leq \frac{|\Gamma(u_x, u_y)|}{|\xi(u_x, u_y)|}\left\|\int_0^t \int_0^a \int_0^b g(t,\tau)\kappa(\omega(u) - \omega(v))\Psi_{n-1}(v_x, v_y; \tau)dv_x dv_y d\tau\right\|$$

$$\leq \frac{\overline{\Gamma}}{\overline{w}}\mathcal{M}\left(\left\|\int_0^a \int_0^b \int_0^a \int_0^b |\kappa(\omega(u) - \omega(v))|^2 du_x du_y dv_x dv_y\right\|\right)^{1/2}$$

$$\times ||max_{0 \leq t \leq T} \int_0^t \left(\int_0^a \int_0^b \{\varphi(v_x, v_y; \tau)\}^2 dv_x dv_y\right)^{1/2} d\tau$$

$$\leq \frac{\overline{\Gamma}}{\overline{w}}\mathcal{M}\mathcal{C}T||\Psi_{n-1}(u_x, u_y; t)||.$$

Using the condition (i) and (ii) and mathematical induction method, we obtain

$$||\Psi_n(u_x, u_y; t)||_{L_2([0,a]\times[0,b])\times C[0,T]} \leq \sigma^n||\Psi_{n-1}(u_x, u_y; t)||;$$

$\sigma = \frac{\mathcal{M}\mathcal{C}\overline{\Gamma}}{\overline{w}}$. We can say that this bound makes the sequence $\Psi_n(u_x, u_y; t)$ converge, and hence the sequence $\{\varphi_n(u_x, u_y; t)\}$ converges. Hence the infinite series

$$\varphi(u_x,u_y;t)=\sum_{i=0}^{\infty}\Psi_i(u_x,u_y;t),$$

$t\in[0,T]$ is uniformly convergent since the terms $\Psi_i(u_x,u_y;t)$ are dominated by σ^n.

4. The System of Fredholm integral equation

Using quadrature method **(Atkinson, 1997)**, the solution of equation (1) is converted to system of Fredholm integral equations. Here we devide the time $[0,T]$, $0\le t\le T=1$ as $0=t_0<t_1<\ldots<t_l<\ldots<t_N=T$, where $t=t_l, l=0,1,\ldots,N$, such that the equation (1) will take the form as:

$$\xi(u_x,u_y)\varphi(u_x,u_y;t_l)=\mathcal{F}(u_x,u_y;t_l)+\Gamma(u_x,u_y)\int_0^{t_l}\int_0^a\int_0^b g(t_l,\tau)\kappa(\omega(u)-\omega(v))\varphi(v_x,v_y;\tau)dv_xdv_yd\tau. \tag{7}$$

The term of the Volterra integral as follows:

$$\int_0^{t_l}\int_0^a\int_0^b g(t_l,\tau)\kappa(\omega(u)-\omega(v))\varphi(v_x,v_y;\tau)dv_xdv_yd\tau=\sum_{j=0}^{l}a_jg(t_l,t_j)\int_0^a\int_0^b\kappa(\omega(u)-\omega(v))\varphi(v_x,v_y)dv_xdv_y+O(h_l^{p+1}), \tag{8}$$

where $h_l\to0$ as $p>0$. Define $h_j=t_{j+1}-t_j$ and $h=\max_{0\le j\le l}h_j$.

The value of p and u_j; depend upon the number of derivatives $g(t,\tau)$ with respect to time t. Now neglecting $O(h_l^{p+1})$ and using value of (8) in (7) we have

$$\xi(u_x,u_y)\varphi(u_x,u_y;t_l)=\mathcal{F}(u_x,u_y;t_l)+\Gamma(u_x,u_y)\sum_{j=0}^{l}a_jg(t_l,t_j)\int_0^a\int_0^b\kappa(\omega(u)-\omega(v))\varphi(v_x,v_y)dv_xdv_y. \tag{9}$$

Using the notations as $\varphi(u_x,u_y;t_l)=\varphi_l(u_x,u_y)$, $\mathcal{F}(u_x,u_y;t_l)=\mathcal{F}_l(u_x,u_y)$, $g(t_l,t_j)=g_{l,j}$.
Then rewriting the equation we have the following form

$$\xi(u_x,u_y)\varphi_l(u_x,u_y)=\mathcal{F}_l(u_x,u_y)+\Gamma(u_x,u_y)\sum_{j=0}^{l}a_jg_{l,j}\int_0^a\int_0^b\kappa(\omega(u)-\omega(v))\varphi_j(v_x,v_y)dv_xdv_y. \tag{10}$$

5. Chebyshev Polynomials

In this part, we use the Chebysev polynomial technique **(Abdel-Aty, 2022)** to find the solution of the integral system (10). Here, we use the extended version of Chebyshev polynomials in $2+1$ dimensional system to solve the algebraic integral system (10), which give rise to consider replacing the given $\kappa(\omega(u)-\omega(v))$ equivalently with a kernel $\kappa_n(\omega(u)-\omega(v))$ that should fulfill the condition

$$\left(\int_0^a\int_0^b\int_0^a\int_0^b |\kappa(\omega(u)-\omega(v))-\kappa_n(\omega(u)-\omega(v))|^2 du_x du_y dv_x dv_y\right)^{1/2} \to 0; n\to\infty. \tag{11}$$

Then, (10) can be expressed in the following mentioned form of an algebraic system

$$\xi(u_x,u_y)\varphi_l^n(u_x,u_y) = \mathcal{F}_l(u_x,u_y)+\Gamma(u_x,u_y)\sum_{j=0}^{l} a_j g_{l,j}\int_0^a\int_0^b \kappa_n(\omega(u)-\omega(v))\,\varphi_j^n(v_x,v_y)dv_x dv_y+\mathcal{R}_n. \tag{12}$$

As a result, the estimated error can be estimated using the following equation:

$$\mathcal{R}_n = |\varphi_l(u_x,u_y)-\varphi_l^n(u_x,u_y)| \to 0; n\to\infty. \tag{13}$$

By writing the kernel of (12) in the following form for using the spectral relationships:

$$\kappa_n(\omega(u_x,u_y)-\omega(v_x,v_y)) = \sum_{r=0}^{q}\sum_{k=0}^{n} T_r(\frac{2u_x}{a}-1)T_k(\frac{2u_y}{b}-1)T_r(\frac{2v_x}{a}-1)T_k(\frac{2v_y}{b}-1), \tag{14}$$

where $T_i(\frac{2u}{a}-1)$ is considered as the Chebyshev polynomial of the first kind and degree i. Now using (14) in (12), we have

$$\begin{aligned}\xi(u_x,u_y)\varphi_l^n(u_x,u_y) = \mathcal{F}_l(u_x,u_y) + \Gamma(u_x,u_y)\sum_{j=0}^{l}\sum_{r=0}^{q}\sum_{k=0}^{n} a_j g_{l,j} T_r(\frac{2u_x}{a}-1)T_k(\frac{2u_y}{b}-1)\\ \int_0^a\int_0^b T_r(\frac{2v_x}{a}-1)T_k(\frac{2v_y}{b}-1)\varphi_j^n(v_x,v_y)dv_x dv_y+\mathcal{R}_n.\end{aligned} \tag{15}$$

The algebraic integral system (15) may be solved numerically. If we put the $l=0$ in the system (15), then we get the value of $\varphi_0^n(u)$ as

$$\varphi_0^n(u_x,u_y) = \frac{\mathcal{F}_0(u_x,u_y)}{\xi(u_x,u_y)} + \frac{\Gamma(u_x,u_y)}{\xi(u_x,u_y)}\sum_{r=0}^{q}\sum_{k=0}^{n} a_0 g_{0,0} T_r(\frac{2u_x}{a}-1)T_k(\frac{2u_y}{b}-1) \int_0^a\int_0^b T_r(\frac{2v_x}{a}-1)T_k(\frac{2v_y}{b}-1)\varphi_0^n(v_x,v_y)dv_x dv_y + \mathcal{R}_n. \tag{16}$$

Now, in the original form, i.e., the equation (15). Here we consider the unknown function $\varphi_l^n(u_x,u_y)$ as

$$\varphi_l^n(u_x,u_y) = A(u_x)A(u_y)B(u_x)B(u_y); A(u_x) = (1-(\frac{2u_x}{a}-1)^2)^{-\frac{1}{2}}. \tag{17}$$

Here $A(u)$ is the weight function of $T_i(u)$, and $B(u)$ is unknown function. Consequently, we have

$$\varphi_{l,\mathrm{p}}^n(u_x,u_y) = \sum_{i=0}^{\mathrm{p}} \Omega_{i,l}\frac{T_i(\frac{2u_x}{a}-1)T_i(\frac{2u_y}{b}-1)}{\sqrt{1-(\frac{2u_x}{a}-1)^2}\sqrt{1-(\frac{2u_y}{b}-1)^2}}. \tag{18}$$

Here $T_i(u)$ are the first kind of Chebyshev polynomials, and $\Omega_{i,l}$ are constants. Again the function $\mathcal{F}_l(u_x,u_y)$ can be approximated as

$$\mathcal{F}_{l,p}(u_x,u_y) = \sum_{i=0}^{\mathrm{p}} \mathfrak{N}_{i,l}\frac{T_i(\frac{2u_x}{a}-1)T_i(\frac{2u_y}{b}-1)}{\sqrt{1-(\frac{2u_x}{a}-1)^2}\sqrt{1-(\frac{2u_y}{b}-1)^2}}, \tag{19}$$

where the coefficients $\mathfrak{N}_{i,l}; i \geq 0$ representing

$$\mathfrak{N}_{i,l} = \begin{cases} \frac{4}{\pi^2}\int_0^a\int_0^b \mathcal{F}_l(u_x,u_y)\frac{T_i(\frac{2u_x}{a}-1)T_i(\frac{2u_y}{b}-1)}{\sqrt{1-(\frac{2u_x}{a}-1)^2}\sqrt{1-(\frac{2u_y}{b}-1)^2}}du_x du_y, & i \neq 0 \\ \frac{2}{\pi^2}\frac{T_i(\frac{2u_x}{a}-1)T_i(\frac{2u_y}{b}-1)}{\sqrt{1-(\frac{2u_x}{a}-1)^2}\sqrt{1-(\frac{2u_y}{b}-1)^2}}; & i=0. \end{cases}$$

By usage of equations (18) and (19) in equation (15), we have

$$\xi(u_x,u_y)\sum_{i=0}^{\mathfrak{p}}\Omega_{i,l}\frac{T_i(\frac{2u_x}{a}-1)T_i(\frac{2u_y}{b}-1)}{\sqrt{1-(\frac{2u_x}{a}-1)^2}\sqrt{1-(\frac{2u_y}{b}-1)^2}}$$
$$=\sum_{i=0}^{\mathfrak{p}}\mathfrak{R}_{i,l}\frac{T_i(\frac{2u_x}{a}-1)T_i(\frac{2u_y}{b}-1)}{\sqrt{1-(\frac{2u_x}{a}-1)^2}\sqrt{1-(\frac{2u_y}{b}-1)^2}} \tag{20}$$
$$+\Gamma(u_x,u_y)\sum_{j=0}^{l}\sum_{r=0}^{q}\sum_{k=0}^{n}a_j g_{l,j}T_r(\frac{2u_x}{a}-1)T_k(\frac{2u_y}{b}-1)$$
$$\int_0^a\int_0^b T_r(\frac{2v_x}{a}-1)T_k(\frac{2v_y}{b}-1)\varphi_j^n(v_x,v_y)dv_xdv_y+\mathcal{R}_n,$$

which fulfill the orthogonal relation as mentioned below

$$\int_0^a\int_0^b\frac{T_n(\frac{2u_x}{a}-1)T_p(\frac{2u_y}{b}-1)T_m(\frac{2u_x}{b}-1)T_q(\frac{2u_y}{b}-1)}{\sqrt{1-(\frac{2u_x}{a}-1)^2}\sqrt{1-(\frac{2u_y}{b}-1)^2}}du_xdu_y$$
$$=\begin{cases}0, if\ n\neq p\ or\ m\neq q\\ \frac{\pi^2}{4}, if\ n=p\neq 0\ and\ m=q\neq 0\\ \frac{\pi^2}{2}, if\ n=0\ and\ m=q\neq 0\ or\ m=0\ and\ n=p\neq 0\\ \pi^2, if\ n=m=p=q=0.\end{cases}$$

6. Convergence analysis

Here, in this part we will proof that the unique numerical solution of the system is exist under some predefined assumptions. The theorems and lemmas that mentioned below will help to achieve this aim.

6.1. The Existence and Unique Numerical Solution

Lemma 6.1. *The kernel $\kappa_n(\omega(u)-\omega(v))$ with* (15) *condition also satisfies the following condition:*

$$\left(\int_0^a\int_0^b\int_0^a\int_0^b|\kappa_n(\omega(u)-\omega(v))|^2du_xdu_ydv_xdv_y\right)^{1/2}=\mathcal{C},$$

$\forall n>n_0, n_0\in N$, $\mathcal{C}$ *is a constant.*

Proof. We use the given cited formula which will lead to proof the lemma

$$\int_0^a\int_0^b\int_0^a\int_0^b|\kappa_n(\omega(u)-\omega(v))|^2du_xdu_ydv_xdv_y$$
$$\leq\int_0^a\int_0^b\int_0^a\int_0^b|\kappa_n\big(\omega(u)-\omega(v)\big)-\kappa\big(\omega(u)-\omega(v)\big)+\kappa\big(\omega(u)-\omega(v)\big)|^2du_xdu_ydv_xdv_y.$$

Then, we get

$$\left(\int_0^a\int_0^b\int_0^a\int_0^b |\kappa_n(\omega(u)-\omega(v))|^2 du_x du_y dv_x dv_y\right)^{1/2}$$
$$\leq \left(\int_0^a\int_0^b\int_0^a\int_0^b \left(\left|\kappa\big(\omega(u)-\omega(v)\big)-\kappa_n\big(\omega(u)-\omega(v)\big)\right| + \left|\kappa\big(\omega(u)-\omega(v)\big)\right|\right)^2 du_x du_y dv_x dv_y\right)^{1/2}$$

and applying condition (11), we get

$\forall\alpha > 0$, there exists $n_0 \in N$

$$\left(\int_0^a\int_0^b\int_0^a\int_0^b |\kappa(\omega(u)-\omega(v))-\kappa_n(\omega(u)-\omega(v))|^2 du_x du_y dv_x dv_y\right)^{1/2} < \alpha,$$

$\forall n > n_0$.

Using Minkowski's inequality and assumption (i), we obtain

$\forall\alpha > 0$, there exists $n_0 \in N$

$$\left(\int_0^a\int_0^b\int_0^a\int_0^b |\kappa_n(\omega(u)-\omega(v))|^2 du_x du_y dv_x dv_y\right)^{1/2} = \mathcal{C};$$

$\forall n > n_0, n_0 \in N$.

Theorem 6.2. *Consider the* Lemma 6.1 *and the assumptions of the* Theorem 2.1 *are satisfied; then the sequence of operators* $\overline{S_n}$ *defined*

$$\overline{S_n}\varphi(u_x,u_y;t) = \frac{\mathcal{F}(u_x,u_y;t)}{\xi(u_x,u_y)} + \frac{\Gamma(u_x,u_y)}{\xi(u_x,u_y)} S_n\varphi(u_x,u_y;t), (\xi(u_x,u_y)\neq 0, \Gamma(u_x,u_y)\neq 0),$$

where

$$S_n\varphi(u_x,u_y;t) = \int_0^t\int_0^a\int_0^b g(t,\tau)\kappa_n(w(u)-w(v))\varphi(v_x,v_y;\tau)dv_x dv_y d\tau. \quad (21)$$

Then, there exists a unique solution if

$$\overline{\Gamma}\mathcal{M}\mathcal{C}T < \overline{w}.$$

To show the existence of unique solution of (21), we have to consider two lemmas, which are as follows

Lemma 6.3. *Under the assumptions* (i)-(iv) *of the* Theorem 2.1, *and in the space* $L_2([0,a]\times[0,b])\times C[0,T], T < 1$, *the sequence of operators* $\overline{S}_n$ *maps the space into itself.*

Proof. Using the assumptions we taken (i) and (iii) and equation (11) and (21), we have

$$
\begin{aligned}
||\overline{S_n}\varphi(u_x,u_y;t)|| &= \left\|\frac{\mathcal{F}(u_x,u_y;t)}{\xi(u_x,u_y)}+\frac{\Gamma(u_x,u_y)}{\xi(u_x,u_y)}S\varphi(u_x,u_y;t)\right\| \\
&\le \frac{\mathcal{G}}{|w(u_x,u_y)|} \\
&+\frac{|\Gamma(u_x,u_y)|}{|\xi(u_x,u_y)|}\left\|\int_0^t\int_0^a\int_0^b g(t,\tau)\kappa_n(\omega(u)-\omega(v))\varphi(v_x,v_y;\tau)dv_xdv_yd\tau\right\| \\
&\le \frac{\mathcal{G}}{\overline{\overline{w}}}+\frac{\overline{\Gamma}}{\overline{\overline{w}}}\mathcal{M}\left(\left\|\int_0^a\int_0^b\int_0^a\int_0^b |\kappa_n(\omega(u)-\omega(v))|^2du_xdu_ydv_xdv_y\right\|\right)^{1/2} \\
&\times max_{0\le t\le T}\int_0^t\left(\int_0^a\int_0^b\{\varphi(v_x,v_y;\tau)\}^2dv_xdv_y\right)^{1/2}d\tau \\
&\le \frac{\mathcal{G}}{\overline{\overline{w}}}+\frac{\overline{\Gamma}}{\overline{\overline{w}}}\mathcal{MC}T||\varphi(u_x,u_y;t)|| \\
&\le \frac{\mathcal{G}}{\overline{\overline{w}}}+\sigma\left\|\varphi(u_x,u_y;t)\right\|, (\sigma=\frac{\overline{\Gamma}}{\overline{\overline{w}}}\mathcal{MC}T)
\end{aligned}
$$

The inequality (2.3), the operator $\overline{S_n}$ maps the ball into itself

$$
\varrho=\frac{\mathcal{G}}{[\overline{w}-\overline{\Gamma}\mathcal{MC}T]}.
$$

Since $\varrho>0,\mathcal{G}>0$, therefore we have $\sigma<1$. Then, we have

$$
\|S_n\varphi(u_x,u_y;t)\|=\|S_n\|\|\varphi(u_x,u_y;t)\|\le\sigma\|\varphi(u_x,u_y;t)\|.
$$

Lemma 6.4. If assumptions (i), (ii) and (iv) are met, then the integral operator $\overline{S}_n$ is continuous in the space $L_2([0,a]\times[0,b])\times C[0,T]$.

Proof. Consider two functions $\varphi_1(u_x,u_y;t)$ and $\varphi_2(u_x,u_y;t)$ in the space $L_2([0,a]\times[0,b])\times C[0,T]$, then we have

$$
\begin{aligned}
&||\overline{S_n}\varphi_1(u_x,u_y;t)-\overline{S_n}\varphi_2(u_x,u_y;t)|| \\
&\le\frac{|\Gamma(u_x,u_y)|}{|\xi(u_x,u_y)|}\left\|\int_0^t\int_0^a\int_0^b|g(t,\tau)||\kappa_n(\omega(u)-\omega(v))||\varphi_1(v_x,v_y;\tau)-\varphi_2(v_x,v_y;\tau)|dv_xdv_yd\tau\right\|.
\end{aligned}
$$

Taking note of assumptions (ii) and (iv), we have

$$
\begin{aligned}
&||\overline{S_n}\varphi_1(u_x,u_y;t)-\overline{S_n}\varphi_2(u_x,u_y;t)|| \\
&\le\frac{\overline{\Gamma}}{\overline{\overline{w}}}\mathcal{M}\left\|\int_0^t\int_0^a\int_0^b|\kappa_n(\omega(u)-\omega(v))||\varphi_1(v_x,v_y;\tau)-\varphi_2(v_x,v_y;\tau)|dv_xdv_yd\tau\right\|.
\end{aligned}
$$

Applying Hölder inequality to the integral term, and then using (i), we finally obtain

$$\left\|\overline{S}_n\varphi_1(u_x,u_y;t)-\overline{S}_n\varphi_2(u_x,u_y;t)\right\|\le\sigma\left\|\varphi_1\left(v_x,v_y;\tau\right)-\ \varphi_2\left(v_x,v_y;\tau\right)\right\|.$$

$\forall n>n_0$. With this inequality, we can see that the operator $\overline{S}_n$ is continuous in $L_2([0,a]\times[0,b])\times C[0,T]$ and then $\overline{S}_n$ is a contraction operator under $\sigma<1$.

6.2. Error Analysis of Numerical Solution

Consider the approximate solution to satisfy the integral equation

$$\xi(u_x,u_y)\varphi_l^n(u_x,u_y)=\mathcal{F}_l(u_x,u_y)+\Gamma(u_x,u_y)\sum_{j=0}^{l}a_j g_{l,j}\int_0^a\int_0^b\kappa_n(\omega(u)-\omega(v))\varphi_j^n(v_x,v_y)dv_x dv_y.$$

Then, the error is $\mathcal{R}_n=[\varphi_l(u_x,u_y)-\varphi_l^n(u_x,u_y)]$.

Let us assume the following assumptions in order to discuss the error

(a) The kernel of position satisfies the discontinuity condition

$$\left(\int_0^a\int_0^b\int_0^a\int_0^b\left|\kappa\big(\omega(u)-\omega(v)\big)-\kappa_n\big(\omega(u)-\omega(v)\big)\right|^2 du_x du_y dv_x dv_y\right)^{1/2}$$

$$=\left(\int_0^a\int_0^b\int_0^a\int_0^b\left|\kappa_{n+1}\big(\omega(u)-\omega(v)\big)\right|^2 du_x du_y dv_x dv_y\right)^{1/2}\le\mathcal{C}^*.$$

(b) The time kernel fulfils $\sum_{j=0}^{n}a_j^2 g_{l,j}^2\le\mathcal{M}^*$,

(c) The unknown function $\varphi(u_x,u_y;t)$, satisfies the condition:

$$\max_l\int_0^a\int_0^b\left|\varphi_l(v_x,v_y)\ -\ \varphi_l^n(v_x,v_y)\right|dv_x dv_y=\ \max_l\int_0^a\int_0^b\left|\varphi_l^{n+1}(v_x,v_y)\right|dv_x dv_y\ =\mathcal{Q}^*.$$

Theorem 6.5. *With the above mentioned assumptions* (a)-(c) *the error of the equation* (1) *is stable with the condition*

$$||\mathcal{R}_n||\le\frac{\overline{\Gamma}}{\overline{S}}\mathcal{M}^*(\mathcal{C}+\mathcal{C}^*)\mathcal{Q}^*.$$

Proof. We have the error

$$\mathcal{R}_n = [\varphi_l(u_x,u_y) - \varphi_l^n(u_x,u_y)]$$
$$= \frac{\Gamma(u_x,u_y)}{\xi(u_x,u_y)} \sum_{j=0}^{l} a_j g_{l,j} \int_0^a \int_0^b [\kappa(\omega(u)-\omega(v))\varphi_j(v_x,v_y) - \kappa_n(\omega(u)-\omega(v))\varphi_j^n(v_x,v_y)]dv_x dv_y$$
$$= \frac{\Gamma(u_x,u_y)}{\xi(u_x,u_y)} \sum_{j=0}^{l} a_j g_{l,j} \int_0^a \int_0^b [\kappa(\omega(u)-\omega(v))\varphi_j(v_x,v_y) - \kappa_n(\omega(u)-\omega(v))\varphi_j^n(v_x,v_y)$$
$$+\kappa(\omega(u)-\omega(v))\varphi_j^n(v_x,v_y) - \kappa_n(\omega(u)-\omega(v))\varphi_j^n(v_x,v_y)]dv_x dv_y.$$

Using the properties of the norms we have

$$||\mathcal{R}_n|| \le \frac{|\Gamma(u_x,u_y)|}{|\xi(u_x,u_y)|}[||\sum_{j=0}^{l} a_j g_{l,j} \int_0^a \int_0^b [\kappa(\omega(u)-\omega(v))(\varphi_j(v_x,v_y) - \varphi_j^n(v_x,v_y))]dv_x dv_y||$$
$$+||\sum_{j=0}^{l} a_j g_{l,j} \int_0^a \int_0^b |\kappa(\omega(u)-\omega(v)) - \kappa_n(\omega(u)-\omega(v))|\varphi_j^n(v_x,v_y)dv_x dv_y||]$$
$$\le \frac{\overline{\Gamma}}{\overline{\overline{w}}}[\sum_{j=0}^{l} a_j^2 g_{l,j}^2] \times [\{||\int_0^a \int_0^b \int_0^a \int_0^b |\kappa(\omega(u)-\omega(v))|^2 du_x du_y dv_x dv_y||]^{1/2}$$
$$\times ||max_l \int_0^a \int_0^b |\varphi_l(v_x,v_y) - \varphi_l^n(v_x,v_y)|dv_x dv_y||\}$$
$$+\{||\int_0^a \int_0^b \int_0^a \int_0^b |\kappa_{n+1}(w(u)-w(v))|^2 du_x du_y dv_x dv_y]^{1/2}$$
$$\times max_l \int_0^a \int_0^b |\varphi_l^{n+1}(v_x,v_y)|dv_x dv_y\}]$$
$$\le \frac{\overline{\Gamma}}{\overline{\overline{w}}}\mathcal{M}^*[(\mathcal{C}\mathcal{Q}^* + \mathcal{C}^*\mathcal{Q}^*)]$$
$$\le \frac{\overline{\Gamma}}{\overline{\overline{w}}}\mathcal{M}^*(\mathcal{C} + \mathcal{C}^*)\mathcal{Q}^*.$$

7. Conclusion

In our present work, QMIE is discussed with singular kernel in a dimension, where we discussed both position and time. We used the quadratic method which convert the QMIE of time and position to a system of Fredholm integral equation such a way that it will be easier to discuss numerical method for this type of equation. In our future work we can study this kind of equation by taking different kernel in same dimension. Another future work of our article that we can discuss by taking singular kernel in integro differential equation.

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