Research Article

Some properties of $\mathcal{I}_3$-λ-statistical cluster points

Verda Gürdal[a] (iD), Ömer Kişi[b,*] (iD)

[a] Department of Mathematics, Suleyman Demirel University, Isparta, Turkey.
[b] Department of Mathematics, Bartın University, Bartın, Turkey.

Abstract: In this note, we investigate some problems concerning the set of λ-statistical cluster points of triple sequences via ideals in finite dimensional spaces, and some of its properties in finite dimensional Banach spaces are proved.

Keywords: λ-statistical convergence, Ideal convergence, Triple sequence.

I. Introduction and preliminaries

Fast (Fast, 1951) proposed statistical convergence as an expansion of the conventional idea of sequential limit. Mursaleen and Edely (Mursaleen and Edely, 2003) and Şahiner et al. (Şahiner et al., 2007) recently presented the idea of statistical convergence for multiple sequences, and several papers on the statistical and ideal convergence of double and triple sequences have been published in (Demirci and Gürdal, 2022; Esi and Savaş, 2015; Huban and Gürdal, 2021; Huban et al., 2020). The idea is based on the notion of natural density of subsets of $\mathbb{N}$, the set of all positive integers which is defined as follows: The natural density of a subset A of $\mathbb{N}$ denoted as $\delta(A)$ is defined by $\delta(A) = \lim_{n \to \infty} \frac{1}{n} |\{k \le n : k \in A\}|$. Generalizing the concept of statistical convergence, Kostyrko et al. introduced the idea of $\mathcal{I}$-convergence in (Kostyrko et al., 2000). More research in this area, as well as applications of ideals, may be found in (Başar, 2021; Das et al., 2014; Gürdal, 2004; Gürdal and Huban, 2014; Gürdal and Şahiner, 2008; Mohiuddine et al., 2017; Nabiev et al., 2007; Pehlivan et al., 2004;

Pehlivan et al., 2006; Yaying and Hazarika, 2020). In another path, the authors of (Das et al., 2011) presented a new sort of convergence known as $\mathcal{I}$ -statistical convergence.

The mathematical characteristics of triple sequences will be investigated further in this study. Section 2 investigates several issues related to the set of λ-statistical cluster points in terms of triple sequences via ideals in finite dimensional Banach spaces. Natural inclusion theorems will be taught in addition to these definitions.

First, we'll go through some fundamental concepts linked to our findings.

The notion of statistical convergence is based on the asymptotic density of the subsets of the set $\mathbb{N}$ of positive integers. Let K be a subset of the set of natural numbers $\mathbb{N}$. We denote by K_n the number of elements of the set K which are less or equal to $n \in \mathbb{N}$. Also $|K_n|$ denotes the cardinality of the set K_n. The natural (asymptotic) density of K is defined by

$$\delta(K) = \lim_{n \to \infty} \frac{1}{n} |K_n|$$

whenever the limit exists. It is said that a sequence $x = (x_k)_{k \in \mathbb{N}}$ is statistically convergent to a point L if for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0.$$

In this case, we write $st - \lim x_k = L$ and S denotes the set of all statistically convergent sequences.

A number $L \in \mathbb{R}$ is called a statistical limit point of a sequence $x = (x_k)$ if there is a set $\{k_1 < k_2 < \cdots k_n < \cdots\} \subseteq \mathbb{N}$, the asymptotic density of which is not zero (i.e., it is greater than zero or does not exist), such that $\lim_{n \to \infty} x_{k_n} = L$. Let Λ_x denote the set of statistical limit points of x. L is an ordinary limit point of a sequence $x = (x_k)$ if there is a subsequence of x that converges to L, and L_x denotes the set of ordinary limit points of x. Let X be a finite-dimensional Banach space, let $x = (x_k)$ be an X -valued sequence, and $\mu \in X$. The sequence (x_k) is norm statistically convergent to μ provided that $\delta(\{k : \|x_k - \mu\| \geq \varepsilon\}) = 0$ for all $\varepsilon > 0$ (see (Connor et al., 2000). Let C be any closed subset of X. Let $d(C, \mu)$ stand for the distance from a point μ to the closed set C, where $d(C, \mu) = \min_{c \in C} \|c - \mu\|$. We formulate the definition of statistical cluster point and some of its properties proved in (Fridy, 1993). Let X be a finite-dimensional Banach space, let $x = (x_k)$ be an X -valued sequence. A point $L \in X$ is called a statistical cluster point if for every $\varepsilon > 0$

$$\lim_{n \to \infty} sup \, n^{-1} |\{k \in \mathbb{N} : \|x_k - L\| < \varepsilon\}| > 0.$$

_____

We will denote the set of statistical cluster points of the sequence x by Γ_x. It is clear that $\Gamma_x \subseteq L_x$ for every

sequence x. For example, we define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} 1, & \text{if } k \text{ is prime,} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\Gamma_x = \{0\}$ and $L_x = \{0,1\}$. So, $\delta(\{k: d(\Gamma_x, x_k) \geq \varepsilon\}) = 0$. Note that if the sequence x. has a bounded

nonthin subsequence, then the set Γ_x is nonempty.

The idea of λ-statistical convergence of sequences $x = (x_k)$ of real numbers has been studied by

Mursaleen (Mursaleen, 2000). Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers tending to

infinity such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$ and $I_n = [n - \lambda_n + 1, n]$. If

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n: |x_k - L| \geq \varepsilon\}| = \lim_n \frac{1}{\lambda_n} |\{k \in (n - \lambda_n + 1, n]: |x_k - L| \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$, then x is said to be λ-statistically convergent to L. In this case, we write $x_k \longrightarrow L(S_\lambda)$ and S_λ denotes all λ-statistically convergent sequences. We can give λ-density of a set $B \subseteq \mathbb{N}$ by $\delta_\lambda(B) = \lim_n \frac{1}{\lambda_n} |\{k \in I_n: k \in B\}|$ whenever the limit exists. The definition of λ-density of a set gives natural density in case $\lambda_n = n$.

The notion of statistical convergence was further generalized in the paper (Kostyrko et al., 2000) using

the notion of an ideal of subsets of the set $\mathbb{N}$. We say that a non-empty family of sets $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is an ideal on $\mathbb{N}$

if $\mathcal{I}$ is hereditary (i.e. $B \subset A \in \mathcal{I} \Rightarrow B \in \mathcal{I}$) and additive (i.e. $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$). An ideal $\mathcal{I}$ on $\mathbb{N}$

for which $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$ is called a non-trivial ideal. A non-trivial ideal $\mathcal{I}$ is called admissible if $\mathcal{I}$ contains all finite

subsets of $\mathbb{N}$. If not otherwise stated in the sequel $\mathcal{I}$ will denote an admissible ideal. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be any ideal.

A class $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N}: \exists A \in \mathcal{I}: M = \mathbb{N} \setminus A\}$ called the filter associated with the ideal $\mathcal{I}$, is a filter on $\mathbb{N}$.

Definition 1.1: Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$ and $x = (x_k)$ be a real sequence. We say that the sequence x is $\mathcal{I}$-convergent to L $\in \mathbb{R}$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N}: |x_k - L| \geq \varepsilon\} \in \mathcal{I}$.

Take for $\mathcal{I}$ the class $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is a non-trivial admissible ideal and $\mathcal{I}_f$-convergence coincides with the usual convergence. For more information about $\mathcal{I}$-convergent, see the study (Nabiev et al., 2007) and references cited therein.

We also recall that the concept of $\mathcal{I}$-statistically convergent was studied in (Das et al., 2011). A sequence (x_k) is said to be $\mathcal{I}$ −statistically convergent to L if for each $\varepsilon > 0$ and $\delta > 0$.

$$\left\{ n \in \mathbb{N}: \frac{1}{n} |\{k \leq n: |x_k - L| \geq \varepsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

In this case, L is called $\mathcal{I}$ −statistical limit of the sequence (x_k) and we write $\mathcal{I} - st - \lim_{k \to \infty} x_k = L$.

_____

We now recall the following basic concepts from (Şahiner et al., 2007; Şahiner and Tripathy, 2008; Subramanian and Esi, 2018) which will be needed throughout the paper.

A function $x: \mathbb{N}^3 \rightarrow \mathbb{R}$ (*or* $\mathbb{C}$) is called a real (or complex) triple sequence. A triple sequence $\left(x_{jkl}\right)$ is said to be convergent to L in Pringsheim's sense if for every > 0, , there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $\left|x_{jkl} - L\right| < \varepsilon$ whenever $j, k, l \geq n_0$. A triple sequence $\left(x_{jkl}\right)$ is said to be bounded if there exists $M > 0$ such that $\left|x_{jkl}\right| < M$ for all $j, k, l \in \mathbb{N}$. We denote the space of all bounded triple sequences by ℓ_∞^3.

Definition 1.2: Let $\mathcal{I}_3$ be an admissible ideal on $2^{\mathbb{N}^3}$ then a triple sequence x_{jkl} is said to be $\mathcal{I}_3 -$convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\left\{(j, k, l) \in \mathbb{N}^3 : \left|x_{jkl} - L\right| \geq \varepsilon\right\} \in \mathcal{I}_3$$

In this case, one writes $\mathcal{I}_3 - \lim x_{jkl} = L$.

Remark 1.3: (i) Let $\mathcal{I}_3(f)$ be the family of all finite subsets of $\mathbb{N}^3$. Then $\mathcal{I}_3(f)$ convergence coincides with the convergence of triple sequences in (Şahiner et al., 2007).

(ii) Let $\mathcal{I}_3(\delta) = \{A \subset \mathbb{N}^3 : \delta(A) = 0\}$. Then $\mathcal{I}_3(\delta)$ convergence coincides with the statistical convergence in (Şahiner et al., 2007).

Example 1.4: Let $\mathcal{I} = \mathcal{I}_3(\delta)$. Define the triple sequence $\left(x_{jkl}\right)$ by

$$\left(x_{jkl}\right) = \begin{cases} 1, & j, k, l \text{ } are \text{ } cubes \\ 5, & otherwise. \end{cases}$$

Then for every $\varepsilon > 0$

$$\delta\left(\left\{(j, k, l) \in \mathbb{N}^3 : \left|x_{jkl} - 4\right| \geq \varepsilon\right\}\right) \leq \lim_{p,q,r} \frac{\sqrt{p}\sqrt{q}\sqrt{r}}{pqr} = 0.$$

This implies that $\mathcal{I} - \lim x_{jkl} = 5$. But, the triple sequence $\left(x_{jkl}\right)$ is not convergent to 5.

Definition 1.5: (Esi, 2013) Let $\lambda = (\lambda_n)$, $\mu = (\mu_m)$ and $v = (v_o)$ be a three non-decreasing sequences of positive real numbers tending to ∞ such that

$$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$$

$$\mu_{m+1} \leq \mu_m + 1, \mu_1 = 1$$

and

$$v_{o+1} \leq v_o + 1, v_1 = 1$$

Let $K \subseteq \mathbb{N}^3$. The number

$$\delta_{\lambda_3}(K) = P - \lim_{n,m,o \to \infty} \frac{1}{\lambda_{n,m,o}} |\{j \in I_n, k \in J_m, l \in L_o : (j,k,l) \in K\}|$$

is said to the λ_3-density of K, provided the limit exists, where $\lambda_{n,m,o} = \lambda_n \mu_m v_o$ and $I_n = [n - \lambda_n + 1, n]$, $J_m = [m - \mu_m + 1, m]$ and $L_o = [o - v_o + 1, o]$ for $n, m, o = 1, 2, \ldots$

Definition 1.6: (Esi, 2013) A triple sequence $x = (x_{jkl})$ is said to be triple λ_3-statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{n,m,o \to \infty} \frac{1}{\lambda_{n,m,o}} |\{j \in I_n, k \in J_m, l \in L_o : |x_{jkl} - L| \geq \varepsilon\}| = 0.$$

In this case we write $S_{\lambda_3} - \lim x = \ell$ or $x_{jkl} \to \ell(S_{\lambda_3})$. If $\lambda_{n,m,o} = nmo$, for all n, m, o then the notion of S_{λ_3}-statistically convergent sequence reduces the concept of statistical convergence for triple sequences in (Şahiner et al., 2007).

Throughout the chapter we consider the ideals of $2^{\mathbb{N}}$ by $\mathcal{I}$; the ideals of $2^{\mathbb{N} \times \mathbb{N}}$ by $\mathcal{I}_2$ and the ideals of $2^{\mathbb{N}^3}$ by $\mathcal{I}_3$. Throughout this section, let X be a finite-dimensional Banach space.

II. Main Results

In this section, we give some properties of the set of $\mathcal{I}_3 - \lambda$-statistical cluster point use of triple sequences in finite dimensional Banach spaces. First we define the concept of $\mathcal{I}_3 - \lambda$-statistically convergence.

Definition 2.1: A real triple sequence $x = (x_{jkl})$ in X is $\mathcal{I}_3 - \lambda$-statistically convergent to $\mu \in X$ if for each $\varepsilon > 0$,

$$\mathcal{I}_3 - \lim_{n,m,o \to \infty} \frac{1}{\lambda_{n,m,o}} |\{j \in I_n, k \in J_m, l \in L_o : |x_{jkl} - \mu| \geq \varepsilon\}| = 0,$$

where $\lambda_{n,m,o} = \lambda_n \mu_m v_o$ and $I_n = [n - \lambda_n + 1, n]$, $J_m = [m - \mu_m + 1, m]$ and $L_o = [o - v_o + 1, o]$ for $n, m, o = 1, 2, \ldots$

Example 2.2: Let $\mathcal{I}_3 = \mathcal{I}_3(f) = \{A \subseteq \mathbb{N}^3 : \delta(A) = 0\}$. Define the triple sequence (x_{jkl}) by

$$x_{jkl} = \begin{cases} (j,k,l), & \text{for } n - \sqrt{\lambda_n} + 1 \le j \le n, m - \sqrt{\mu_m} + 1 \le k \le m, \\ & o - \sqrt{\nu_o} + 1 \le j \le o, \\ 3, & \text{otherwise.} \end{cases}$$

It is easy to see that, this sequence is $\mathfrak{I}_3 - \lambda$-statistically convergent to 3.

Definition 2.3: For a real triple sequence $x = (x_{jkl})$ the point $\mu \in X$ is called $\mathfrak{I}_3 - \lambda$-statistical cluster point if for every $\varepsilon > 0$

$$\mathfrak{I}_3 - \text{limsup}_{n,m,o\to\infty} \frac{1}{\lambda_{n,m,o}} \left| \left\{ j \in I_n, k \in J_m, l \in L_o : |x_{jkl} - \mu| < \varepsilon \right\} \right| = 0.$$

By $\Gamma_x^{\mathfrak{I}_3}(\lambda)$ we denote the set of all $\mathfrak{I}_3 - \lambda$-statistical cluster points of the triple sequence x.

Lemma 2.4: Let $x = (x_{jkl})$ be a triple sequence in X and $\Gamma_x^{\mathfrak{I}_3}(\lambda) \ne \emptyset$. Then $\Gamma_x^{\mathfrak{I}_3}(\lambda)$ is a closed set.

Proof: Assume that $\mu_{n,m,o} \in \Gamma_x^{\mathfrak{I}_3}(\lambda)$ and $\mu_{n,m,o} \to \mu$. Then, there exist numbers n', m', o' such that $\|\mu_{n',m',o'} - \mu\| < \frac{\varepsilon}{2}$ for a given positive number $\varepsilon > 0$. Since $\mu_{n',m',o'} \in \Gamma_x^{\mathfrak{I}_3}(\lambda)$ by Definition 2.3, we have

$$\mathfrak{I}_3 - \text{limsup}_{n,m,o\to\infty} \frac{1}{\lambda_{n,m,o}} \left| \left\{ j \in I_n, k \in J_m, l \in L_o : |x_{jkl} - \mu_{n',m',o'}| < \frac{\varepsilon}{2} \right\} \right| = 0.$$

which implies

$$\left\{ j \in I_n, k \in J_m, l \in L_o : |x_{jkl} - \mu_{n',m',o'}| < \frac{\varepsilon}{2} \right\} \subset \left\{ j \in I_n, k \in J_m, l \in L_o : |x_{jkl} - \mu| < \varepsilon \right\}.$$

Hence,

$$\mathfrak{I}_3 - \limsup_{n,m,o\to\infty} \frac{1}{\lambda_{n,m,o}} \left| \left\{ j \in I_n, k \in J_m, l \in L_o : |x_{jkl} - \mu| < \varepsilon \right\} \right| > 0$$

and $\mu \in \Gamma_x^{\mathfrak{I}_3}(\lambda)$.

Lemma 2.5: Let $x = (x_{jkl})$ be a triple sequence in X and $\Gamma_x^{\mathfrak{I}_3}(\lambda)$ be a set of $\mathfrak{I}_3 - \lambda$-statistical cluster points of the triple sequence x. Let Z be a non-empty compact subset of X. If

$$\mathfrak{I}_3 - \limsup_{n,m,o\to\infty} \frac{1}{\lambda_{n,m,o}} \left| \left\{ j \in I_n, k \in J_m, l \in L_o : x_{jkl} \in Z \right\} \right| > 0, \qquad (2.1)$$

then $\Gamma_x^{\mathfrak{I}_3}(\theta_3) \cap Z \ne \emptyset$.

Proof: We suppose that $\Gamma_x^{\mathfrak{I}_3}(\theta_3) \cap Z \neq \emptyset$. In this case no point $\mu \in Z$ is $\mathfrak{I}_3 - \lambda$-statistical cluster point, that is for each $\mu \in Z$ there is a positive number $\varepsilon = \varepsilon(\mu) > 0$ such that

$$\mathfrak{I}_3 - \limsup_{n,m,o\to\infty} \frac{1}{\lambda_{n,m,o}} \left|\left\{j \in I_n, k \in J_m, l \in L_o: \|x_{jkl} - \mu\| < \varepsilon\right\}\right| = 0.$$

Let $D_\varepsilon(\mu) = \{z \in X: \|z - \mu\| < \varepsilon\}$. Then, the sets $D_{\varepsilon_{prs}}(\mu_{prs})$ form an open covering of Z, and because Z is compact, there are finitely many sub covering of sets $D_{\varepsilon_{prs}}(\mu_{prs})$, $p, r, s = 1, 2, \dots, z$ such that $Z \subset \cup_{p,r,s=1}^z D_{\varepsilon_{prs}}(\mu_{prs})$. It is clear that

$$\mathfrak{I}_3 - \limsup_{n,m,o\to\infty} \frac{1}{\lambda_{n,m,o}} \left|\left\{j \in I_n, k \in J_m, l \in L_o: \|x_{jkl} - \mu_{prs}\| < \varepsilon_{prs}\right\}\right| = 0$$

for every p, r, s. Then

$$\begin{aligned}
\left|\left\{j \in I_n, k \in J_m, l \in L_o: x_{jkl} \in Z\right\}\right| & \\
\leq \sum_{p=1}^z \sum_{r=1}^z \sum_{s=1}^z &\left|\left\{j \in I_n, k \in J_m, l \in L_o: \|x_{jkl} - \mu_{prs}\|\right.\right. \\
& \left.\left. < \varepsilon_{prs}\right\}\right|,
\end{aligned} \tag{2.2}$$

and from (2.2) we obtain

$$\mathfrak{I}_3 - \limsup_{n,m,o\to\infty} \frac{1}{\lambda_{n,m,o}} \left|\left\{j \in I_n, k \in J_m, l \in L_o: x_{jkl} \in Z\right\}\right| = 0$$

which contradicts (2.1).

Theorem 2.6: Let $x = (x_{jkl})$ be a triple sequence in X. Then, for every $\varepsilon > 0$

$$\mathfrak{I}_3 - \lim_{n,m,o\to\infty} \frac{1}{\lambda_{n,m,o}} \left|\left\{j \in I_n, k \in J_m, l \in L_o: \rho\left(\Gamma_x^{\mathfrak{I}_3}(\lambda), x_{jkl}\right) \geq \varepsilon\right\}\right| = 0.$$

Proof: Let x_{jkl} be an $\mathfrak{I}_3 -$bounded triple sequence. Then, we can find a closed set $Z \subset X$ such that $x_{jkl} \in Z$ for every j, k, l. We will denote the $\varepsilon -$neighbourhood of Z by $D_\varepsilon(Z) = \{z \in X: \rho(Z, z) < \varepsilon\}$. Now we assume that

$$\mathfrak{I}_3 - \limsup_{n,m,o\to\infty} \frac{1}{\lambda_{n,m,o}} \left|\left\{j \in I_n, k \in J_m, l \in L_o: \rho\left(\Gamma_x^{\mathfrak{I}_3}(\theta_3), x_{jkl}\right) \geq \varepsilon\right\}\right| > 0.$$

Then for the set $\check{Z} = Z \setminus \dot{N}_\varepsilon\left(\Gamma_x^{\mathfrak{I}_3}(\lambda)\right)$ we have

$$\mathfrak{I}_3 - \limsup_{n,m,o\to\infty} \frac{1}{\lambda_{n,m,o}} \left|\left\{j \in I_n, k \in J_m, l \in L_o: x_{jkl} \in \check{Z} \subset X\right\}\right| > 0.$$

_____

By Lemma 2.4, $\Gamma_x^{\mathcal{I}_3}(\lambda) \cap \check{Z} \neq \emptyset$, which is a contradiction. The proof is complete.

Corollary 2.7: Let x_{jkl} be a triple sequence and $Z \subset X$ be a compact set such that $\Gamma_x^{\mathcal{I}_3}(\lambda) \cap Z \neq \emptyset$. Then for every point $\mu \in Z$ there is $\varepsilon = \varepsilon(\mu) > 0$ with

$$\mathcal{I}_3 - \lim_{n,m,o\to\infty} \frac{1}{\lambda_{n,m,o}} \left| \{ j \in I_n, k \in J_m, l \in L_o : x_{jkl} \in Z \} \right| > 0.$$

Proof: Let $D_\varepsilon(\mu) = \{ y \in X : \|y - \mu\| < \varepsilon \}$. The open sets $D_\varepsilon(\mu), \mu \in Z$ form an open covering of Z. But Z is a compact set and so there exists a finite sub cover of Z, say $D_{m,n,o} = D_{\varepsilon_{m,n,o}}(\mu_{mno}), m, n, o = 1,2, \dots, N$. Clearly, $Z \subset \bigcup_{m,n,o} D_{m,n,o}$ and

$$\mathcal{I}_3 - \lim_{n,m,o\to\infty} \frac{1}{\lambda_{n,m,o}} \left| \{ j \in I_n, k \in J_m, l \in L_o : \|x_{jkl} - \mu_{mno}\| < \varepsilon_{m,n,o} \} \right| = 0$$

for every m, n, o. We have

$$\left| \{ j \in I_n, k \in J_m, l \in L_o : x_{jkl} \in Z \} \right|$$
$$\leq \sum_{m=1}^{N} \sum_{n=1}^{N} \sum_{o=1}^{N} \left| \{ j \in I_n, k \in J_m, l \in L_o : \|x_{jkl} - \mu_{mno}\| < \varepsilon_{m,n,o} \} \right|$$

and therefore

$$\mathcal{I}_3 - \lim_{n,m,o\to\infty} \frac{1}{\lambda_{n,m,o}} \left| \{ j \in I_n, k \in J_m, l \in L_o : x_{jkl} \in Z \} \right|$$
$$\leq \sum_{m=1}^{N} \sum_{n=1}^{N} \sum_{o=1}^{N} \mathcal{I}_3 - \lim_{n,m,o\to\infty} \frac{1}{\lambda_{n,m,o}} \left| \{ j \in I_n, k \in J_m, l \in L_o : \|x_{jkl} - \mu_{mno}\| < \varepsilon_{m,n,o} \} \right| = 0.$$

Then the desired result has been obtained.

References

F. Başar. (2012). Summability theory and its applications. Bentham Science Publishers, İstanbul.

Connor, J., Ganichev, M., Kadets, V. (2000). A Characterization of Banach Spaces with separable duals via weak statistical convergence. Journal of Mathematical Analysis and Applications, 244, 251-261.

Das, P., Dutta, S., Mohiuddine, S.A., Alotaibi, A. (2014). A-statistical cluster points in finite dimensional spaces and application to turnpike theorem. Abstract and Applied Analysis, Volume 2014, Article ID 354846, 7 pages.

Das, P., Savaş, E., Ghosal, S.K. (2011). On generalizations of certain summability methods using ideals. Applied Mathematics Letters, 24 (9), 1509-1514.

Demirci, I.A., Gürdal, M. (2022). On Lacunary Statistical $\widetilde{\phi}$-convergence for Triple Sequences of Sets Via Ideals. Journal of Applied Mathematics and Informatics, 40 (3-4), 433-444.

Esi, A. (2013). λ_3-statistical convergence of triple sequences on probabilistic normed space. Global Journal of Mathematical Analysis, 1 (2), 29-36.

Esi, A., Savaş, E. (2015). On lacunary statistically convergent triple sequences in probabilistic normed space. Applied Mathematics & Information Sciences, 9 (5), 2529-2534.

Fast, H. (1951). Sur la convergence statistique. Colloquium Mathematicae, 2, 241-244.

Fridy, J.A. (1993). Statistical limit points. Proceedings of the American Mathematical Society, 4, 1187-1192.

Gürdal, M. (2004). Some type of convergence. Doctoral Dissertation, Sleyman Demirel University, Isparta.

Gürdal, M., Huban, M.B. (2014). On $\mathcal{I}$-convergence of double sequences in the topology induced by random 2-norms. Matematicki Vesnik, 66 (1), 73-83.

Gürdal, M., Şahiner, A. (2008). Extremal $\mathcal{I}$ -limit points of double sequences. Applied Mathematics E-Notes, 8, 131-137.

Huban, M.B., Gürdal, M. (2021). Wijsman lacunary invariant statistical convergence for triple sequences via Orlicz function. Journal of Classical Analysis, 17 (2), 119-128.

Huban, M.B., Gürdal, M., Savaş, E. (2020). $\mathcal{I}$-statistical limit superior and $\mathcal{I}$ -statistical limit inferior of triple sequences. 7th International Conference on Recent Advances in Pure and Applied Mathematics (ICRAPAM2020), Proceeding Book of ICRAPAM, 42-49.

Kostyrko, P., Macaj, M., Šalát, T. (2000). $\mathcal{I}$ -Convergence. Real Analysis Exchange, 26 (2), 669-686.

Mohiuddine, S.A., Hazarika, B., Alotaibi, A. (2017). On statistical convergence of double sequences of fuzzy valued functions. Journal of Intelligent & Fuzzy Systems, 32 (6), 4331-4342.

Mursaleen, M. (2000). λ-statistical convergence. Mathematica Slovaca, 50, 111-115.

Mursaleen, M., Edely, O.H.H. (2003). Statistical convergence of double sequences. Journal of Mathematical Analysis and Applications, 288, 223-231.

Nabiev, A., Pehlivan, S., Gürdal, M. (2007). On $\mathcal{I}$-Cauchy sequences. Taiwanese Journal Mathematics, 11 (2), 569-576.

Pehlivan, S., Güncan, A., Mamedov, M.A. (2004). Statistical cluster points of sequences in finite dimensional spaces. Czechoslovak Mathematical Journal, 54 (1), 95-102.

_____

Pehlivan, S., Gürdal, M., Fisher, B. (2006). Lacunary statistical cluster points of sequences. Math. Commun., 11, 39-46.

Şahiner, A., Gürdal, M., Düden, F.K. (2007). Triple sequences and their statistical convergence. Selçuk Journal of Applied Mathematics, 8 (2), 49-55.

Şahiner, A., Tripathy, B.C. (2008). Some $\mathcal{I}$-related properties of triple sequences. Selçuk Journal of Applied Mathematics, 9 (2), 9-18.

Subramanian, N., Esi, A. (2018). Wijsman rough lacunary statistical convergence on $\mathcal{I}$-Cesàro triple sequences. International Journal of Analysis and Applications, 16 (5), 643-653.

Yaying, T., Hazarika, B. (2020). Lacunary arithmetic statistical convergence. National Academy Science Letters, 43, 547-551.