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Research Article

Statistical convergence within octonion metric structures

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Abstract: This paper investigates statistical convergence and completeness within the framework of octonion-valued metric spaces (OVMSs). By equipping the algebra of octonions with a suitable partial order, we extend classical notions of convergence, Cauchy sequences, and statistical density to the non-associative setting of octonions. Several fundamental properties are established, highlighting the interaction between statistical convergence and completeness in OVMSs. Our findings show that statistical convergence implies statistical Cauchy behaviour in these spaces, and conditions under which statistical completeness is achieved are clarified. The study not only generalizes conventional metric spaces but also reveals the structural impact of octonions' non-associativity on convergence theory. Potential implications for both pure mathematics and applied areas such as physics, control theory, and artificial intelligence are discussed.

Keywords: Statistical convergence, octonion metric spaces, Cauchy sequences, non-associative algebras.

MSC 2020: 40A05, 11B05, 11N05, 11A99

1. Introduction

Following Hamilton's seminal 1843 discovery of quaternions, Graves formulated octonions, an extension of higher-dimensional number systems. Later, Arthur Cayley independently refined and advanced the concept. This progression, from real numbers through complex numbers and quaternions to octonions, constitutes a structured growth of hypercomplex number systems, governed by the Cayley-Dickson process. The method gradually expands the dimension, beginning with real numbers in one dimension, advancing to complex numbers in two dimensions, then to quaternions in four dimensions, and finally to octonions in eight dimensions. Each step in this sequence reveals increasingly intricate algebraic structures and properties, paving the way for novel mathematical insights and applications.

Octonions stand out in this progression due to their distinctive mathematical characteristics. Real and complex numbers satisfy commutativity, while quaternions break this property but remain associative. Octonions, however, deviate significantly from this pattern, exhibiting neither commutativity nor associativity. Their non-associative nature implies that the order of grouping terms affects the result of multiplication, as seen in $(ab)c \neq a(bc)$. This deviation, while precluding them from fitting into standard algebraic frameworks, places them within the broader category of alternative algebras. These algebras adhere to weaker forms of associativity, encapsulated by the Moufang identities.

The Cayley-Dickson construction, through which octonions are derived from quaternions, is fundamental to defining their unique multiplication rules. These rules are often illustrated using the Fano plane, a diagram that visually encodes the relationships between the basis elements of the octonion space. This representation is a valuable tool in mathematical applications, providing a clear depiction of how the basis elements interact during multiplication and highlighting the intricate structure and properties of octonions.

While long considered of primarily theoretical significance, the inherently non-associative algebraic structure offers practical utility in contexts involving the interaction of high-dimensional data. As highlighted in the work of **(Kansu et al., 2020)**, octonionic frameworks have been employed in formulating field equations for dyons that inherently preserve duality symmetry. These field expressions, structurally akin to Maxwell's equations, effectively model the interplay between electric and magnetic fields. Owing to their eight-dimensional composition, octonions serve as a natural medium for encoding intricate interdependencies between electromagnetic components within a single formalism.

In machine learning, octonions have proven effective for handling high-dimensional data. **Wu et al. (2020)** proposed deep octonion networks (DONs), which utilize the compact algebraic structure of octonions to fuse multi-dimensional features across neural network layers. Within this framework, octonions enable efficient data representation and processing, with tasks like image classification demonstrating improved performance and convergence.

Furthermore, **Takahashi et al. (2021)** extended the use of octonions to control systems, specifically for the dynamic control of robot manipulators. In this context, octonion-valued neural networks capture both spatial and temporal dynamics, with their non-associative property allowing the network to model complex multi-axis movements necessary for precise manipulator control.

Although octonions possess a non-commutative and non-associative nature that poses initial challenges, these structures have facilitated innovative uses in fields such as modern theoretical physics, control systems, and artificial intelligence, where managing adaptive representations and multi-dimensional data is crucial.

For in-depth information on octonions, their subalgebraic structures, and interdisciplinary applications, one can refer to the works by **(Albert, 1934; Baez, 2001; Conway and Smith, 2005; Dray and Manogue, 2015; Okubo, 1995)**.

The investigation of sequence convergence and summability has constituted a foundational branch of pure mathematics, with its theoretical advancements influencing diverse domains such as applied mathematics, computational modelling, computer science, topology, functional and Fourier analysis, and measure theory. Among various convergence notions, statistical convergence has garnered growing attention in recent decades. Originally introduced by **Fast (1951)**, this concept has since undergone extensive development, with numerous researchers exploring its theoretical implications and practical applications across multiple mathematical disciplines. Recently, several authors have investigated various aspects of statistical convergence and its generalizations in different settings (see, e.g., (**Belen & Mohiuddine, 2013**; **Mohiuddine, 2016**; **Mohiuddine & Alamri, 2019**; **Mursaleen & Mohiuddine, 2014)**). These studies provide significant motivation for extending the notion of convergence to more generalized algebraic structures such as octonion-valued metric spaces.

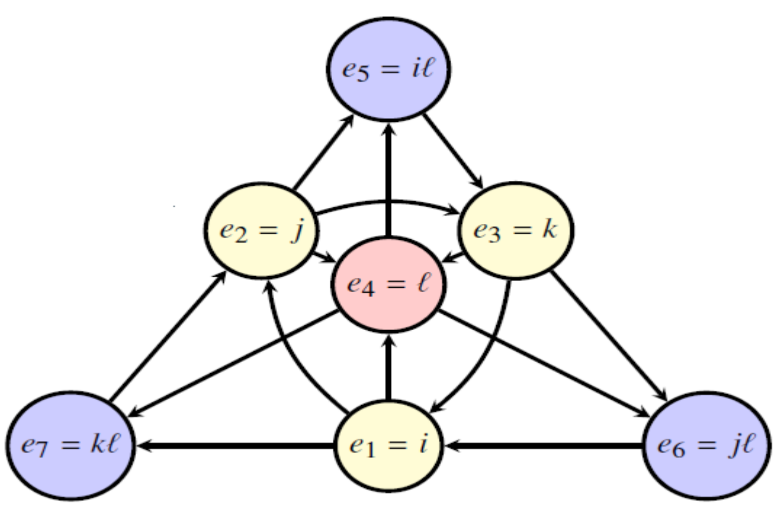
In particular, **Abazari (2022)** defined statistically convergent sequences according to metrics on generalised metric spaces and investigated the fundamental properties of this form of statistical convergence. **Gürdal and Şahiner (2012)** contributed by introducing statistical convergence concepts for sequences in 2-Banach spaces, statistical convergence and order for linear operators and elements, conditions for the statistical convergence and stability of linear operators, and some applications. **Gürdal and Yamancı (2015)** focused on the relationship between statistical convergence and Berezin symbols to solve some problems in operator theory. **Li et al. (2015)** introduced statistical convergence in conic metric spaces, discussed statistically ordered compact spaces, and investigated their behaviour. **Savaş et al. (2022)** investigated the behaviour of statistically convergent sequences for fuzzy variables in the credibility space, while **Nabiev et al. (2019)** introduced statistically localised sequences in metric spaces and studied their fundamental properties, and **Yamancı and Gürdal (2016)** proposed the concept of discrete statistical Borel convergence, providing characterisations of Schatten--von Neumann class operators via Berezin symbols. Later on it was further investigated rom the sequence space point of view and linked with summability theory by **Aral et al. (2024)**, **Gürdal and Kişi (2024), Indumathi et al. (2023), Kişi (2023), Kolancı and Gürdal (2023)** and many others.

By introducing a suitable partial order on the algebra of octonions, this study develops and formalizes key analytical concepts adapted to this non-associative setting, including statistical convergence, statistical Cauchy sequences, and the notion of statistical density for subsequences. These concepts are generalized within the framework of octonion-valued metric spaces (hereafter referred to as OVMSs), allowing us to explore their properties and the connections between them. This study is also motivated by prior work such as **Çetin et al. (2025)** and **Kişi et al. (2025)** which examined convergence and ideal convergence in octonion-valued settings. Moreover, we explore the impact of octonions' non-associative nature on statistical convergence and related sequence properties. This includes investigations into statistical Cauchy sequences and the characteristics of statistically dense subsequences. OVMSs extend usual metric spaces by leveraging the rich and complex algebraic properties of octonions, which introduce a higher-dimensional and non-associative framework. Unlike conventional vector spaces or rings, octonions lack the multiplicative associativity property, making them particularly intriguing when applied in these defined metric spaces. The distinctive algebraic structure of octonions offers not only an enriched perspective on convergence behavior but also introduces new mathematical challenges and opportunities for further exploration.

2. Literature

We now turn our attention to the octonion algebra O, which constitutes a non-associative generalization of the classical quaternion division algebra.

The construction of O proceeds by extending the quaternionic basis $1, i, j, k$ with additional imaginary units, beginning with ℓ, to form a complete eight-dimensional basis. The resulting algebraic framework, including its multiplication rules and illustrative representations, is developed in alignment with the exposition provided in **(Fiorenza et al., 2021)**.



Thus, any element $\mathfrak{x} \in O$ can be represented as:

$$\mathfrak{x} = o_0 + o_1e_1 + o_2e_2 + o_3e_3 + o_4e_4 + o_5e_5 + o_6e_6 + o_7e_7, \quad o_n \in \mathbb{R}, \quad \text{where } n = 0, \dots, 7.$$

A standard basis for the octonion algebra O is given by the set $1, e_1, e_2, e_3, e_4, e_5, e_6, e_7$, comprising one real unit and seven distinct imaginary units. The comprehensive multiplication rules for these basis elements are presented in the table in the study **(Kişi et al., 2025; Çetin et al., 2025)**.

Octonions can be expressed as an ordered tuple of eight real components $(o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7)$ where addition is performed coordinate-wise and multiplication follows the rules outlined in a designated multiplication table. The initial component o_0 is referred to as the real part, whereas the subsequent seven components $(o_1, o_2, o_3, o_4, o_5, o_6, o_7)$ make up the imaginary part. Therefore, as previously mentioned, every quaternion may be expressed as $(o_0, \overleftrightarrow{u})$,where $\overleftrightarrow{u} = (o_1, o_2, o_3, o_4, o_5, o_6, o_7)$ represents the imaginary components and o_0 denotes the real part. From this representation, the following properties become readily apparent:

$$\begin{aligned} \mathfrak{x} &:= (o_0, \overleftrightarrow{u}), \quad \overleftrightarrow{u} \in \mathbb{R}^7; \quad o_0 \in \mathbb{R} \\ &= (o_0, (o_1, o_2, o_3, o_4, o_5, o_6, o_7)); \quad o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7 \in \mathbb{R} \\ &= o_0 + o_1e_1 + o_2e_2 + o_3e_3 + o_4e_4 + o_5e_5 + o_6e_6 + o_7e_7. \end{aligned}$$

We now introduce a partial ordering relation $\preccurlyeq$ on the octonion algebra O, taking into account its inherent non-associativity and non-commutativity, defined as follows.

$$\mathfrak{x} \preccurlyeq \mathfrak{x}' \text{ iff } Re(\mathfrak{x}) \leq \mathrm{Re}(\mathfrak{x}'), \mathrm{Im}_e(\mathfrak{x}) \leq \mathrm{Im}_e(\mathfrak{x}'), \ \mathfrak{x}, \mathfrak{x}' \in H; \ e = e_1, e_2, e_3, e_4, e_5, e_6, e_7,$$

where $\mathrm{Im}_{e_n} = o_n; n = 1, \dots, 7$.

To verify that $\mathfrak{x} \preccurlyeq \mathfrak{x}'$, it suffices that at least one of the 256 conditions---derived from all possible combination sums between 0 to 8 is satisfied. A detailed analysis of the aforementioned 256 conditions shows that OVMSs can be defined as a generalization of complex metric spaces, originally introduced by **Azam et al. (2011)**, by considering the codomain to be the field of complex numbers.

Definition 2.1. **(Azam et al., 2011; Hadzic & Gajic, 1986)** *Given a non-empty set S. If the transformation $\Omega_C: S \times S \longmapsto C$ defined on this set meets the subsequent criteria:*

- $0_C \preccurlyeq \Omega_C(s,t)$, *for any $s, t \in S$ and $\Omega_C(s,t) = 0_C \Leftrightarrow s = t$.*
- $\Omega_C(s,t) = \Omega_C(t,s)$ *for any $s, t \in S$.*
- $\Omega_C(s,t) \preccurlyeq \Omega_C(s,v) + \Omega_C(v,t)$ *for all $s, t, v \in S$.*

Then the pair (S, Ω_C) is said to be a complex metric space.

These are further extended to metric spaces with values in the quaternionic skew field, as described by **Ahmed et al. (2014)**. In this context, quaternions provide a non-commutative generalization of such metric spaces, allowing their application within the framework of Clifford algebra analysis.

Definition 2.2. **(Ahmed et al., 2014)** *Let S be a nonempty set. A transformation $\Omega_C: S \times S \longmapsto H$ is said to satisfy the following conditions if it meets the criteria listed below.*

- $0_H \preccurlyeq \Omega_H(s,t)$ *for any* $s,t \in S$ *and* $\Omega_H(s,t) = 0_H \Leftrightarrow s = t$,
- $\Omega_H(s,t) = \Omega_H(t,s)$ *for any* $s,t \in S$,
- $\Omega_H(s,t) \preccurlyeq \Omega_H(s,v) + \Omega_H(v,t)$ *for every* $s,t,v \in S$.

In this case Ω_H is referred to as a metric taking values in the quaternions on the set S, and the structure (S,Ω_H) is known as a quaternion-valued metric space.

Next, we introduce OVMSs, which provide a noteworthy extension of traditional metric spaces, distinguished by their lack of commutativity and associativity.

The definitions, examples, theorems, and propositions in this section are taken from **(Çetin et al., 2025; Kişi et al., 2025)**.

Definition 2.3. **(Çetin et al., 2025; Kişi et al., 2025)** *Let S be a nonempty set. A mapping $\Omega_O: S \times S \longmapsto O$ is called an octonion-valued metric on S if it satisfies the following properties:*

- $0_O \preccurlyeq \Omega_O(s,t)$ *for every* $s,t \in S$ *and* $\Omega_O(s,t) = 0_O$ *iff* $s = t$,
- $\Omega_O(s,t) = \Omega_O(t,s)$ *for every* $s,t \in S$,
- $\Omega_O(s,t) \preccurlyeq \Omega_O(s,v) + \Omega_O(v,t)$ *for every* $s,t,v \in S$.

If these conditions hold, then the pair (S,Ω_O) is called an OVMS.

Example 2.4. *Let $\Omega_O: S \times S \longmapsto O$ be an octonion-valued function defined by* $\Omega_O(\mathfrak{x},\mathfrak{x}') = |o_0 - o_0'| + |o_1 - o_1'|e_1 + |o_2 - o_2'|e_2 + |o_3 - o_3'|e_3 + |o_4 - o_4'|e_4 + |o_5 - o_5'|e_5 + |o_6 - o_6'|e_6 + |o_7 - o_7'|e_7$ *where* $\mathfrak{x},\mathfrak{x}' \in O$ *with*

$$\mathfrak{x} = o_0 + o_1e_1 + o_2e_2 + o_3e_3 + o_4e_4 + o_5e_5 + o_6e_6 + o_7e_7,$$

$$\mathfrak{x}' = o_0' + o_1'e_1 + o_2'e_2 + o_3'e_3 + o_4'e_4 + o_5'e_5 + o_6'e_6 + o_7'e_7;$$

$$o_i, o_i' \in \mathbb{R};\ \ i = 0,1,2,3,4,5,6,7.$$

Consequently, the pair (O,Ω_O) constitutes an OVMS.

Here, we present an example of an octonion-valued metric constructed on a domain that does not belong to the class of conventional numerical sets.

Example 2.5. *Consider the set* $X = \{u, v, w\}$ *consisting of three arbitrary elements. The distances between these elements are specified as follows:*

$$\Omega_O(u,v) = \Omega_O(v,u) = 3 + 4e_1 - 6e_2 + 4e_3 + 3e_4 + 3e_5 - 2e_6 + e_7$$
$$\Omega_O(v,w) = \Omega_O(w,v) = 1 + 2e_1 + 3e_3 - 5e_4 - 3e_6 + 4e_7$$
$$\Omega_O(u,w) = \Omega_O(w,u) = 2 + 3e_1 + e_2 + e_3 - 2e_4 + 2e_5 - e_6 + 5e_7$$
$$\Omega_O(u,u) = \Omega_O(v,v) = \Omega_O(w,w) = 0 + 0e_1 + 0e_2 + 0e_3 + 0e_4 + 0e_5 + 0e_6 + 0e_7.$$

Given the values

$$\|\Omega_O(u,v)\| = 10, \|\Omega_O(u,w)\| = 7, \|\Omega_O(w,v)\| = 8,$$

$$\|\Omega_O(u,v) + \Omega_O(u,w)\| = \sqrt{195}, \|\Omega_O(u,v) + \Omega_O(v,w)\| = \sqrt{200},$$

$$\|\Omega_O(w,v) + \Omega_O(u,w)\| = \sqrt{169} = 13,$$

it follows from direct computation that the conditions specified in Definition 2.3 *are indeed satisfied.*

3. Main Results

This section introduces several definitions related to OVMSs, based on the framework established in earlier work. The focus is on concepts such as convergent sequences, Cauchy sequences, and bounded sequences, which have been extended through a statistical generalization of the traditional definitions. These notions are applicable when a large portion, rather than the entirety, of the sequence's terms demonstrate properties like convergence, Cauchy behavior, or density.

It is clear from the foregoing definitions and illustrative examples that the proposed framework generalizes the classical concept of a metric in a natural and consistent manner, seamlessly incorporating the complex- and quaternion-valued cases as special instances. To further elucidate the interconnections among these structures, we now present the following propositions.

Proposition 3.1. *Any metric space defined over the quaternions can be naturally embedded into an OVMS, preserving the underlying structure.*

Proposition 3.2. *Each metric space defined over the complex field can be regarded as a special case of both quaternion-valued and OVMSs.*

Proposition 3.3. *Every standard metric space equipped with real-valued distances can be naturally embedded into a complex-valued, quaternion-valued, and OVMS via the well-known algebraic inclusions among the number systems.*

Accordingly, we now move on to introduce several fundamental notions that naturally arise from the preceding definition.

Definition 3.4. *A point $s \in S$ is considered an interior point of the set $A \subset S$ if there exists an octonion $0_O \prec r \in O$ such that*

$$B(s,r) = \{t \in S \;:\; \Omega_O(s,t) \prec r\} \subset A.$$

Definition 3.5. *A point $s \in S$ is defined as a limit point of $A \subset S$ if for any $0_O \prec r \in O$*

$$B(s,r) \cap (A - \{s\}) \neq \emptyset.$$

Definition 3.6. *A set O is considered open if every element of O is an interior point of O. A subset $C \subset S$ is regarded as closed if it contains all its limit points. The family*

$$F = \{B(s,r): \quad s \in S, 0_O \prec r\}$$

constitutes a subbasis for the Hausdorff topology τ defined on the set S.

Now, we will recall the definitions, examples, theorems, and propositions related to the concepts of convergence within the previously defined OVMSs and these unique mathematical structures. Since the theorems and propositions presented here will be given without proofs, interested readers can refer to **(Çetin et al., 2025)** and **(Kişi et al., 2025))** for the corresponding proofs.

Definition 3.7. *Let $s \in S$ and let s_k be a sequence in S. We say that (s_k) converges to s if for every $\mathfrak{x} \in O$ with $0_O \prec \mathfrak{x}$, there exists $k_0 \in \mathbb{N}$ such that for all $k > k_0$,*

$$\Omega_O(s_k, s) \prec \mathfrak{x},$$

In this case (s_k) is called a convergent sequence with limit point s; denoted by

$$s_k \to s \text{ as } k \to \infty, \text{ or simply } \lim_{k\to\infty} s_k = s.$$

Theorem 3.8. *Let (s_k) be a sequence in the OVMS (S, Ω_O). If (s_k) converges to a point $s_0 \in S$, then any arbitrary subsequence (s_{k_n}) also converges, and this subsequence converges to the point s_0.*

Corollary 3.9. *In both the quaternion-valued metric space (S, Ω_H) and the complex-valued metric space (S, Ω_C), every subsequence of a convergent sequence converges to the same point.*

Definition 3.10. *A sequence (s_k) in the octonion-valued metric space (S, Ω_O) is called a Cauchy sequence if for every $\mathfrak{x} \in O$ with $0_O \prec \mathfrak{x}$, there exists $k_0 \in \mathbb{N}$ such that*

$$\Omega_O(s_{k+m}, s_k) \prec \mathfrak{x} \text{ for all } k > k_0 \text{ and } m \in \mathbb{N}.$$

If every Cauchy sequence in (S, Ω_O) converges to a point in S, then the space (S, Ω_O) is said to be complete, and (S, Ω_O) is referred to as a complete octonion-valued metric space.

Note that not every OVMS must be complete. The following example of an OVMS supports this.

Example 3.11. *Let $\Omega_O \; : \; \mathbb{N}^+ \times \mathbb{N}^+ \to O$ be an octonion-valued function defined by*

$$\Omega_0(n,m) = \left|\frac{1}{n} - \frac{1}{m}\right| + \left|\frac{2}{n} - \frac{2}{m}\right| e_1 + \left|\frac{3}{n} - \frac{3}{m}\right| e_2 + \left|\frac{4}{n} - \frac{4}{m}\right| e_3 + \left|\frac{5}{n} - \frac{5}{m}\right| e_4 + \left|\frac{6}{n} - \frac{6}{m}\right| e_5 + \left|\frac{7}{n} - \frac{7}{m}\right| e_6 + \left|\frac{8}{n} - \frac{8}{m}\right| e_7$$

where $n, m \in \mathbb{N}^+$. Then $(\mathbb{N}^+, \Omega_O)$ defines an octonion-valued metric space. Nevertheless, due to the fact that 0 *is not an element of the set of positive natural numbers $\mathbb{N}^+$, the space in question does not satisfy the completeness property.*

Theorem 3.12. *If (s_k) be a Cauchy sequence in the OVMS (S, Ω_O) has the subsequence (s_{k_n}) converges to the point s_0, then Cauchy sequence (s_k) also converges, and this Cauchy sequence converges to the point s_0.*

Corollary 3.13. *In both the quaternion-valued metric space (S, Ω_H) and the complex-valued metric space (S, Ω_C), if an arbitrary Cauchy sequence has convergent subsequence, then the Cauchy sequence converges to the same point.*

Definition 3.14. *A sequence (s_k) in an OVMS (S, Ω_O) is said to be statistically convergent to a point $s \in S$,*

$$s_k \overset{stg}{\to} s,$$

if for every $0_O \prec \mathfrak{x}$, the following condition holds:

$$\lim_{N \to \infty} |\{k \leq N : \Omega_O(s_k, s) \nprec \mathfrak{x}\}| = 0.$$

In this formulation, the quantity

$$|\{k \leq N : \Omega_O(s_k, s) \nprec \mathfrak{x}\}|$$

counts the number of terms in the sequence for which the octonion-valued "distance" to the point s does not fall strictly below the bound $\mathfrak{x}$ with respect to the partial order defined in Definition 2.3. *The statistical convergence criterion requires that the density of such indices tends to zero, i.e.,*

$$\frac{|\{k \leq N : \Omega_O(s_k, s) \nprec \mathfrak{x}\}|}{N} \to 0$$

as $N \to 0$. *This condition is necessary for the sequence* (s_k) *to be statistically convergent to the point* s *in the OVMS.*

In classical convergence, for all $0_O \prec \mathfrak{x} \in O$, there exists $N \in \mathbb{N}$ such that for $k \geq N$, $\Omega_O(s_k, s) \prec \mathfrak{x}$ holds. In statistical convergence, $\Omega_O(s_k, s) \prec \mathfrak{x}$ must hold only for the majority of the terms in the sequence; some terms are allowed to be far from s.

Statistical convergence is a generalized version of the classical convergence concept, indicating that the majority of the terms in a sequence converge to a point. This concept can be considered a weaker form of classical convergence.

Example 3.15. *Let* S *be a nonempty set and let* (s_k) *be a sequence in* S *defined via an octonion-valued metric* Ω_O. *Define the sequence elements through the function* $f: \mathbb{N} \to S$ *as follows:*

$$f(k) = s_k = \begin{cases} s_8, \text{ if } k = n^3 \text{for some } n \in \mathbb{N}, \\ s_2, \text{ otherwise.} \end{cases}$$

Here, the exceptional set $A = \{k \; : \; k = n^3$ *for some* $n \in \mathbb{N}\} \subset \mathbb{N}$. *The asymptotic (natural) density of* A *is given by*

$$\frac{|A \cap \{1,2, \ldots, N\}|}{N} \approx \frac{1}{N^{2/3}},$$

and since $\lim_{N \to \infty} \frac{1}{N^{2/3}} = 0$, *the set of indices has density zero. Therefore, the sequence* (s_k) *is statistically convergent, and its statistical limit is* s_2*; that is,*

$$s_k \overset{stg}{\to} s_2.$$

Theorem 3.16. *In any OVMS* (S, Ω_O), *every convergent sequence is also statistically convergent with respect to the same limit point.*

Proof. According to the notion of convergence given in Definition 3.7, for every $\mathfrak{x} \in O$ with $0_O \prec \mathfrak{x}$, there exists an index $k_0 \in \mathbb{N}$ such that for all $k > k_0$, the inequality

$$\Omega_O(s_k, s) \prec \mathfrak{x}$$

holds. Consequently, the number of terms that fail to satisfy this condition must be finite. Since the asymptotic density of any finite subset of the natural numbers is zero, it follows from Definition 3.14 that the sequence is statistically convergent.

Statistical convergence, unlike standard convergence, requires that the majority of the terms, rather than all of them, are close to s. This concept is particularly significant in the analysis of large data sets and complex structures.

Theorem 3.17. *Given an OVMS (S, Ω_O) and a sequence (s_k) in this space, a necessary and sufficient condition for the sequence (s_k) to converge statistically to s is*

$$\|\Omega_O(s_k, s)\| \overset{stg}{\rightarrow} 0$$

as $k \to \infty$.

Proof. Let the sequence (s_k) statistical converge to point s. Given a real number $\varepsilon > 0$, suppose that

$$o = \frac{\varepsilon}{2\sqrt{2}} + e_1 \frac{\varepsilon}{2\sqrt{2}} + e_2 \frac{\varepsilon}{2\sqrt{2}} + e_3 \frac{\varepsilon}{2\sqrt{2}} + e_4 \frac{\varepsilon}{2\sqrt{2}} + e_5 \frac{\varepsilon}{2\sqrt{2}} + e_6 \frac{\varepsilon}{2\sqrt{2}} + e_7 \frac{\varepsilon}{2\sqrt{2}}.$$

From the definition of statistical convergence, for $\forall 0_O \prec \mathfrak{x}' \in O$, we have

$$\lim_{N\to\infty} \frac{1}{N} |\{k \le N \ : \ \Omega_O(s_k, s) \nprec \mathfrak{x}'\}| = 0.$$

In this case, specially for $0_O \prec \mathfrak{x} \in O$ and there exists

$$\lim_{N\to\infty} \frac{1}{N} |\{k \le N \ : \ \Omega_O(s_n, s) \nprec \mathfrak{x}\}| = \lim_{N\to\infty} \frac{1}{N} |\{k \le N \ : \|\Omega_O(s_k, s)\| \ge \|\mathfrak{x}\| = \varepsilon\}| = 0.$$

Thus, we obtain

$$\lim_{N\to\infty} \frac{1}{N} |\{k \le N \ : \|\Omega_O(s_k, s)\| < \|\mathfrak{x}\| = \varepsilon\}| = 1.$$

Hence, $\|\Omega_O(s_k, s)\| \overset{stg}{\rightarrow} 0$ as $k \to \infty$.

On the other hand, suppose that $\|\Omega_O(s_k, s)\| \overset{stg}{\rightarrow} 0$ as $k \to \infty$. In this case, for a given $\mathfrak{x} \in O$ with $0_O \prec \mathfrak{x}$, there exists a real number $\delta > 0$, such that for any $\mathfrak{x}' \in O$, the following holds:

$$\|\mathfrak{x}'\| < \delta \Rightarrow \mathfrak{x}' \prec \mathfrak{x}.$$

For this δ, we find

$$\lim_{N\to\infty}\frac{1}{N}|\{k\leq N \ : \|\Omega_O(s_k,s)\|\geq\|\mathfrak{x}\|=\varepsilon\geq\delta\geq\|\mathfrak{x}'\|\}|=\lim_{N\to\infty}\frac{1}{N}|\{k\leq N \ : \ \Omega_O(s_k,s)\nprec\mathfrak{x}'=0.\}|$$

This leads to

$$\lim_{N\to\infty}\frac{1}{N}|\{k\leq N \ : \|\Omega_O(s_k,s)\|<\|\mathfrak{x}\|=\varepsilon\}|=1.$$

which implies

$$\lim_{N\to\infty}\frac{1}{N}|\{k\leq N \ : \ \Omega_O(s_k,s)\nprec\mathfrak{x}\}|=0.$$

Hence, the sequence (s_k) converges statistically to point s.

Theorem 3.18. *Let (s_k) be a sequence in the OVMS (S,Ω_O). Let both $s_k\overset{stg}{\to}s_0$ and $s_k\overset{stg}{\to}t_0$ hold in this metric space. In that case, $s_0=t_0$.*

Proof. Assume that $s_k\overset{stg}{\to}s_0$ and $s_k\overset{stg}{\to}t_0$. In connection with this, and by the definition of statistical convergence provided in Definition 3.14, for any $\varepsilon>0$ and for every $\mathfrak{x}\in O$ with $0_O\prec\mathfrak{x}$, consider the case where

$$\mathfrak{x}=\frac{\varepsilon}{4\sqrt{2}}+e_1\frac{\varepsilon}{4\sqrt{2}}+e_2\frac{\varepsilon}{4\sqrt{2}}+e_3\frac{\varepsilon}{4\sqrt{2}}+e_4\frac{\varepsilon}{4\sqrt{2}}+e_5\frac{\varepsilon}{4\sqrt{2}}+e_6\frac{\varepsilon}{4\sqrt{2}}+e_7\frac{\varepsilon}{4\sqrt{2}}.$$

The following equalities hold:

$$\lim_{N\to\infty}\frac{1}{N}|\{k\leq N \ : \ \Omega_O(s_k,s_0)\nprec\mathfrak{x}\}|=0,$$

and

$$\lim_{N\to\infty}\frac{1}{N}|\{k\leq N \ : \ \Omega_O(s_k,t_0)\nprec\mathfrak{x}\}|=0.$$

By applying the triangle inequality (the third axiom) of the OVMS, we deduce that

$$0_O\preccurlyeq\Omega_O(s_0,t_0)\preccurlyeq\Omega_O(s_0,s_k)+\Omega_O(s_k,t_0),$$

and as a result,

$$\lim_{N\to\infty}\frac{1}{N}|\{k\leq N \ : \ \Omega_O(s_k,s_0)+\Omega_O(s_k,t_0)\nprec\mathfrak{x}\}|=0.$$

By the partial ordering property, it follows that

$$0\leq\|\Omega_O(s_0,t_0)\|\leq\|\Omega_O(s_0,s_k)+\Omega_O(s_k,t_0)\|\leq\|\Omega_O(s_0,s_k)\|+\|\Omega_O(s_k,t_0)\|<\varepsilon.$$

From this, we conclude that $\|\Omega_O(s_0,t_0)\| = 0$, which implies $\Omega_O(s_0,t_0) = 0_O$. Finally, invoking the first axiom of the octonion-valued metric space, we conclude that $s_0 = t_0$. This establishes the desired result and completes the proof.

Proposition 3.19. *In both cases the quaternion-valued metric space (S,Ω_H) and the complex-valued metric space (S,Ω_C), the statistical limit is unique.*

Definition 3.20. *Consider a sequence (s_k) in the framework of an OVMS (S,Ω_O). For a sequence (s_k), a subsequence (s_{k_n}) is called a statistical cluster subsequence if:*

$$\forall 0 \prec \mathfrak{x} \in O \text{ such that } \lim_{N\to\infty} \sup \frac{1}{N}\left|\{n \le N: \Omega_O(s_{k_n},s) \nprec \mathfrak{x}\}\right| = 0.$$

Theorem 3.21. *Let (s_k) and (t_k) be two sequences in the OVMS (S,Ω_O). If $t_k \overset{stg}{\rightarrow} s$, and $\Omega_O(s_k,s) \preccurlyeq \Omega_O(t_k,s)$ for each $k \in N$, then $s_k \overset{stg}{\rightarrow} s$.*

Proof. Since $t_k \overset{stg}{\rightarrow} s$, it follows from Theorem 3.17 that

$$\|\Omega_O(t_k,s)\| \overset{stg}{\rightarrow} 0 \quad \text{as} \quad k \to \infty.$$

For each $0_O \prec \mathfrak{x} \in O$ and $k \in \mathbb{N}$, we observe that

$$\{k \le N \ : \ \Omega_O(t_k,s) \prec \mathfrak{x}\} \subseteq \{k \le N \ : \ \Omega_O(s_k,s) \prec \mathfrak{x}\}.$$

Thus,

$$1 = \lim_{N\to\infty} \frac{1}{N}|\{k \le N \ : \ \Omega_O(t_k,s) \prec \mathfrak{x}\}| \le \lim_{N\to\infty} \frac{1}{N}|\{k \le N \ : \ \Omega_O(s_k,s) \prec \mathfrak{x}\}|.$$

Since the asymptotic density can be at most 1, by Definition 3.14, for all $0_O \prec \mathfrak{x}$, we have

$$\lim_{N\to\infty} \frac{1}{N}|\{k \le N \ : \ \Omega_O(s_k,s) \nprec \mathfrak{x}\}| = 0.$$

Consequently, $s_k \overset{stg}{\rightarrow} s$.

Definition 3.22. **(Li et al., 2015)** *A subsequence (s_{k_n}) of a sequence (s_k) is statistically dense in (s_k) if the index set $\{k_n : n \in \mathbb{N}\}$ is a statistically dense subset of $\mathbb{N}$, in other words,*

$$\lim_{N\to\infty} \frac{1}{N}|\{k_n \le N \ : \ n \in \mathbb{N}\}| = 1.$$

Theorem 3.23. *In an OVMS (S,Ω_O), let (s_k) be an arbitrary sequence. In this case, the following conditions are equivalent:*

1) *The sequence (s_k) is statistically convergent in the octonion-valued space (S,Ω_O).*
2) *There exists a sequence (t_k) in S that converges such that $s_k = t_k$ for almost all $k \in \mathbb{N}$.*

3) *The sequence* (s_k) *contains a statistically dense subsequence* (s_{k_n}), *which is a convergent sequence.*

4) *The sequence* (s_k) *contains a statistically dense subsequence* (s_{k_n}), *which is statistically convergent.*

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$**.** Assume that $s_k \overset{stg}{\rightarrow} s$. By Definition 3.14, for all $0_O \prec \mathfrak{x}$, we have

$$\lim_{N\to\infty} \frac{1}{N} |\{k \le N \ : \ \Omega_O(s_k, s) \nprec \mathfrak{x}\}| = 0.$$

Specially, let $\|\mathfrak{x}'\| = 1$ for a chosen element $\mathfrak{x}' \in O$. Then

$$\lim_{N\to\infty} \frac{1}{N} \left|\left\{k \le N \ : \ \Omega_O(s_k, s) \prec \frac{\mathfrak{x}'}{3}\right\}\right| = 1.$$

This implies that there exists $N_1 \in \mathbb{N}$ such that

$$\lim_{N\to\infty} \frac{1}{N} \left|\left\{k \le N \ : \ \Omega_O(s_k, s) \prec \frac{\mathfrak{x}}{3}\right\}\right| > 1 - \frac{1}{3}$$

for every $N > N_1$. A sequence (N_n) of natural numbers can be chosen so that

$$\lim_{N\to\infty} \frac{1}{N} \left|\left\{l \le N \ : \ \Omega_O(s_l, s) \prec \frac{\mathfrak{x}'}{3^n}\right\}\right| > 1 - \frac{1}{3^n}$$

is satisfied whenever for all $N > N_n$. Suppose that $N_n < N_{n+1}$ for each $n \in \mathbb{N}$. Let t_l be defined as

$$t_l = \begin{cases} s_l, & 1 \le l \le N_1, \\ s_l, & N_n < l \le N_{n+1}, \Omega_O(s_l, s) \prec \dfrac{\mathfrak{x}'}{3^n}, \\ s, & \text{otherwise.} \end{cases}$$

Given $0_O \prec \mathfrak{x} \in O$, choose $n \in \mathbb{N}$ such that $\frac{\mathfrak{x}'}{3^n} \prec o$. Then, $\Omega_O(t_l, s) \prec \mathfrak{x}$ for all $l > N_n$, indicating that the sequence (t_l) converges to s.

For all $0_O \prec \mathfrak{x} \in O$, there exists $n \in N$ with $\frac{\mathfrak{x}'}{3^n} < o$. Let $N \in N$. If $N_n < N \le N_{n+1}$, then

$$\{l \le N \ : \ t_l \ne s_l\} \subset \{1,2,..,N\} - \{l \le N \ : \ \Omega_O(s_l, s) \prec \frac{\mathfrak{x}'}{3^n}\},$$

so

$$\frac{1}{N} |\{l \le N \ : \ t_l \ne s_l\}| \le 1 - \frac{1}{N} |\{l \le N \ : \ \Omega_O(s_l, s) \prec \frac{\mathfrak{x}}{3^n}\}| < \frac{1}{3^n} < \|\mathfrak{x}\|.$$

Therefore $\lim_{N\to\infty} \frac{1}{N} |\{l \le N \ : \ t_l \ne s_l\}| = 0$. Hence $s_l = t_l$ for almost every $l \in \mathbb{N}$.

$(2) \Rightarrow (3)$**.** Assume that (t_k) is a convergent sequence in S with $s_k = t_k$ for almost every $k \in \mathbb{N}$. In this situation, $\lim_{N\to\infty} \frac{1}{N} |\{l \le N \ : \ t_l \ne s_l\}| = 0$. If we take $(t_k) = (s_{k_n})$, in this situation, from Definition 3.7 and Definition 3.22, (t_k) is both a convergent sequence and a statistically dense subsequence of (s_k).

$(\mathbf{3}) \Rightarrow (\mathbf{4})$ If we take $(t_k) = (s_{k_n})$ as a subsequence, it can be directly seen from the definition of a statistically dense subsequence (Definition 3.22) and the definition of statistical convergence (Definition 3.14).

$(\mathbf{4}) \Rightarrow (\mathbf{1})$**.** Assume that there exists a statistically dense subsequence (s_{k_n}) of the sequence (s_k) with the sequence (s_{k_n}) is statistically convergent. By the definition of statistical convergence, we have for all $0_O \prec \mathfrak{x}$, we have

$$\lim_{N\to\infty} \frac{1}{N} \left|\left\{k_n \leq N \ : \ \Omega_O(s_{k_n}, s) \nprec \mathfrak{x}\right\}\right| = 0.$$

and let

$$s_{k_n} \overset{stg}{\to} s \ \text{ as } \ n \to \infty.$$

It follows that

$$\lim_{N\to\infty} \frac{1}{N} \left|\left\{k_n \leq N \ : \ \Omega_O(s_{k_n}, s) \prec \mathfrak{x}\right\}\right| = 1.$$

Because it happens that for each $0_O \prec o \in O$,

$$\{k_n \in N \ : \ \Omega_O(s_{k_n}, s) \prec \mathfrak{x}\} \subset \{k \in N \ : \ \Omega_O(s_k, s) \prec \mathfrak{x}\},$$

and

$$\lim_{N\to\infty} \frac{1}{N} |\{k \leq N \ : \ \Omega_O(s_k, s) \prec \mathfrak{x}\}| \geq \lim_{N\to\infty} \frac{1}{N} \left|\left\{k_n \leq N \ : \ \Omega_O(s_{k_n}, s) \prec \mathfrak{x}\right\}\right| = 1.$$

We have

$$s_k \overset{stg}{\to} s \ \text{ as } \ k \to \infty.$$

Corollary 3.24. *In the OVMS* (S, Ω_O)*, every statistically convergent sequence admits a subsequence that converges in the usual sense within the same space.*

Definition 3.25. *A sequence* (s_k) *in an OVMS* (S, Ω_O) *is called a statistical Cauchy sequence if, for every* $\mathfrak{x} \in O$ *with* $0_O \prec \mathfrak{x}$*, there exists an index* $l \in \mathbb{N}^+$ *(possibly depending on the norm of* $\mathfrak{x}$*) such that*

$$\lim_{N\to\infty} \frac{1}{N} \left|\left\{k_n \leq N \ : \ \Omega_O(s_{k_n}, s) \nprec \mathfrak{x}\right\}\right| = 0.$$

If we carefully examine this definition,

$$|\{k_n \leq N \ : \ \Omega_O(s_k, s_l) \nprec \mathfrak{x}\}|$$

represents the number of terms in the sequence (s_k) *in* S *whose octonion value, indicating the distance between the elements of the sequence, does not precede* $\mathfrak{x}$ *according to the partial ordering relation given in* Definition 2.3. *The ratio of these terms to the total number of terms* N *must approach zero as* $N \to \infty$*. In other words,*

$$\frac{|\{k_n \leq N \ : \ \Omega_O(s_k, s_l) \nprec \mathfrak{x}\}|}{N} \to 0$$

as $N \to \infty$. This is a necessary condition for the sequence to be statistically Cauchy.

In accustomed definition Cauchy sequence, for every $0_O \prec \mathfrak{x} \in O$, we have $N \in \mathbb{N}$ with as $k, l \geq N$, $\Omega_O(s_k, s_l) \prec \mathfrak{x}$ satisfies. In statistical Cauchy sequence, $\Omega_O(s_k, s_l) \prec \mathfrak{x}$ must satisfy only for the majority of the terms in the sequence; it is acceptable for the distances between some terms to follow after $\mathfrak{x}$.

The concept of a statistical Cauchy sequence is a generalized version of the classical Cauchy sequence and can be understood as a sequence where the distances between the majority of its terms precede $\mathfrak{x}$ in the ordering.

Theorem 3.26. *Let (S, Ω_O) be an OVMS, and let (s_k) be a sequence in S. Then (s_k) is a statistical Cauchy sequence iff*

$$\|\Omega_O(s_k, s_{k+m})\| \overset{stg}{\to} 0$$

as $k \to \infty$, where the convergence is in the statistical sense.

Proof. We assume that (s_k) is a statistically Cauchy sequence in S. From Definition 3.25, as for all $0_O \prec \mathfrak{x}$. For a given $\mathfrak{x} \in O$, there exists an index $l \in \mathbb{N}^+$ (possibly depending on the norm $|\mathfrak{x}|$) such that

$$\lim_{N\to\infty} \frac{1}{N} |\{k, l \leq N \ : \ \Omega_O(s_k, s_l) \nprec \mathfrak{x}\}| = 0.$$

As a given real number $\varepsilon > 0$, suppose that

$$\mathfrak{x} = \frac{\varepsilon}{2\sqrt{2}} + e_1 \frac{\varepsilon}{2\sqrt{2}} + e_2 \frac{\varepsilon}{2\sqrt{2}} + e_3 \frac{\varepsilon}{2\sqrt{2}} + e_4 \frac{\varepsilon}{2\sqrt{2}} + e_5 \frac{\varepsilon}{2\sqrt{2}} + e_6 \frac{\varepsilon}{2\sqrt{2}} + e_7 \frac{\varepsilon}{2\sqrt{2}}.$$

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} |\{k, l \leq N \ : \ \Omega_O(s_k, s_l) \nprec \mathfrak{x}\}| \\ &= \lim_{N\to\infty} \frac{1}{N} |\{k, l \leq N \ : \|\Omega_O(s_k, s_l)\| \geq \|\mathfrak{x}\| = \varepsilon\}| = 0. \end{aligned}$$

In this context, it follows from Theorem 3.17 and Definition 3.14 that

$$\|\Omega_O(s_k, s_l)\| \overset{stg}{\to} 0 \quad \text{as} \quad k \to \infty.$$

On the other hand, we assume that $\|\Omega_O(s_k, s_{k+m})\| < \|\mathfrak{x}\| \overset{stg}{\to} 0$ as $k \to \infty$. So, given $\mathfrak{x} \in O$ with $0_O \prec \mathfrak{x}$, there is a real number $\delta > 0$ such that as $\mathfrak{x}' \in O$,

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} |\{k \leq N \ : \|\Omega_O(s_k, s_{k+m})\| \geq \|\mathfrak{x}\| = \varepsilon \geq \delta \geq \|\mathfrak{x}'\|\}| \\ &= \lim_{N\to\infty} \frac{1}{N} |\{k \leq N \ : \ \Omega_O(s_k, s_{k+m}) \nprec \mathfrak{x}''\}| = 0. \end{aligned}$$

Corresponding to this δ, there exists $l \in \mathbb{N}^+$ depending on the norm of $\mathfrak{x} \in O$, so, we get

$$\lim_{N\to\infty} \frac{1}{N} |\{k \leq N \ : \|\Omega_O(s_k, s_l)\| < \|\mathfrak{x}\| = \varepsilon\}| = 1.$$

This implies that

$$\lim_{N\to\infty} \frac{1}{N} |\{k \leq N \ : \ \Omega_O(s_k, s) \nprec \mathfrak{x}\}| = 0.$$

Hence the sequence (s_k) is statistical Cauchy sequence. Thus, the proof is complete.

Theorem 3.27. *In an OVMS, statistical convergence of a sequence implies that the sequence is statistical Cauchy.*

Proof. Let (s_k) be a sequence in the OVMS (S, Ω_O), and suppose that $s_k \overset{stg}{\rightarrow} s$. Then, for every $\mathfrak{x} \in O$ with $0_O \prec \mathfrak{x}$, the following holds:

$$\lim_{N\to\infty} \frac{1}{N} |\{k \leq N \ : \ \Omega_O(s_k, s) \nprec \mathfrak{x}\}| = 0.$$

Additionally, by the independence of the representation of statistical convergence and by its definition, for every $0_O \prec \mathfrak{x}'$, there exists a $K \in \mathbb{N}$ such that when $k, l > K$, and given the partial ordering definition above and the fact that $0_O \prec \mathfrak{x}' \in O$, it follows for the octonion $\frac{\mathfrak{x}'}{2}$ that $0_O \prec \frac{\mathfrak{x}'}{2} \in O$. Furthermore,

$$\lim_{N\to\infty} \frac{1}{N} \left|\left\{k \leq N \ : \ \Omega_O(s_k, s) \nprec \frac{\mathfrak{x}''}{2}\right\}\right| = 0.$$

and

$$\lim_{N\to\infty} \frac{1}{N} \left|\left\{k \leq N \ : \ \Omega_O(s_l, s) \nprec \frac{\mathfrak{x}'}{2}\right\}\right| = 0.$$

hold.

Thus, for indices $k, l > K$, it follows from the triangle inequality (the third axiom) of the OVMS that:

$$\Omega_O(s_k, s_l) \preccurlyeq \Omega_O(s_k, s) + \Omega_O(s, s_l) = \mathfrak{x}',$$

so for each $N \in \mathbb{N}$,

$$\{k \leq N \ : \ \Omega_O(s_k, s) \prec \mathfrak{x}''\} \subset \{k, l \leq N \ : \ \Omega_O(s_k, s_l) \prec \mathfrak{x}'\},$$

and

$$\lim_{N\to\infty} \frac{1}{N} |\{k \leq N \ : \ \Omega_O(s_l, s_k) \nprec \mathfrak{x}''\}| = 0.$$

Therefore, since $\Omega_O(s_k, s_l) \prec \mathfrak{x}'$ holds for every $0_O \prec \mathfrak{x}' \in O$, the sequence (s_k) is a statistical Cauchy sequence. The proof is complete.

Proposition 3.28. *In both quaternion-valued and complex-valued metric spaces, every statistically convergent sequence is also a statistical Cauchy sequence.*

Definition 3.29. *If every statistically Cauchy sequence in an OVMS (S, Ω_O) is statistically convergent, then the space (S, Ω_O) is called a statistically complete octonion-valued metric space.*

Corollary 3.30. *Every statistically complete OVMS is a complete.*

Note that not every OVMS must be statistically complete. The following example of an OVMS supports this.

Example 3.31. *Let $d_O : \mathbb{N}^+ \times \mathbb{N}^+ \to O$ be an octonion valued function defined by*

$$d_O(n,m) = \begin{cases} 1_O, \text{ if } m \text{ is prime,} \\ \Omega_O(n,m), \text{ otherwise,} \end{cases}$$

where $n, m \in \mathbb{N}^+$ and $\Omega_O(n,m)$ is defined as in Example 3.11. *Then $(\mathbb{N}^+, d_O)$ defines an octonion valued metric space. However, since it is $0 \notin \mathbb{N}^+$, this OVMS is not statistically complete.*

A statistically dense subsequence is not necessarily statistically Cauchy. Statistical density and statistical Cauchy-ness are distinct concepts, and their relationship depends on the structure of the sequence. Statistical density implies that the sequence clusters around certain points or values, while statistical Cauchy-ness indicates that the distances between terms of the sequence decrease in a controlled manner. However, if a sequence has a statistically dense subsequence, then this subsequence is statistically convergent within the sequence, and thus it is also a statistically Cauchy subsequence.

Remark 3.32. *Every ring forms a module over itself, and every field forms a vector space over itself, as is commonly known. Let's be clear, though, that octonions cannot form a module over themselves since they lack multiplicative associativity, which makes them ineligible even as rings. Because of this, our established metric spaces and the associated conclusions are of special importance.*

4. Conclusion

In this study, statistical convergence and completeness have been systematically examined within the framework of octonion-valued metric spaces. By introducing a partial order on octonions, we were able to extend classical notions of convergence, Cauchy sequences, and statistical density to a non-associative algebraic setting. The analysis has shown that every convergent sequence in an OVMS is also statistically convergent, and that statistical convergence naturally leads to statistical Cauchy behavior. Furthermore, it has been demonstrated that completeness and statistical completeness are not guaranteed properties in OVMSs but instead depend on the structural characteristics of octonions. These findings provide a significant generalization of conventional metric space theory and emphasize the distinctive influence of non-associativity on convergence concepts. Beyond the theoretical framework, the results suggest potential applications in areas such as physics, control theory, and machine learning, where high-dimensional and non-associative structures frequently arise. Future investigations may focus on extending these ideas to other non-associative algebras or exploring concrete applications of OVMSs in modeling complex multidimensional systems.

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