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Research Article

Completeness and Cartesian product in neutrosophic rectangular n-normed spaces

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Abstract: This study introduces the new concept of neutrosophic rectangular n-normed spaces (NR-n-NS), along with essential foundational definitions. We then investigate the Cartesian product of such spaces and examine how this operation influences their structural characteristics, and demonstrate that the Cartesian product of neutrosophic rectangular n-normed spaces retains the same structure. Furthermore, it establishes that the Cartesian product of complete neutrosophic rectangular n-normed spaces is itself complete, and presents several auxiliary results and theorems are provided to support and enrich the theoretical development of these spaces.

Keywords: Neutrosophic n-normed spaces; rectangular n-normed space; neutrosophic rectangular n-normed space; neutrosophic rectangular n-normed space; the Cartesian product of neutrosophic rectangular n-normed spaces.

MSC 2020: 03B52, 46S40, 54A05, 46B20, 03E7.

1. Introduction

In 1986, Atanassov (Atanassov, 1986) introduced the intuitionistic fuzzy set as an extension of classical fuzzy set theory. Building on this framework, Mohammed and Ataa (Mahammad & Ataa, 2014) later developed the structure of an intuitionistic fuzzy topological space and explored its characteristics. Further developments occurred in 2020 when Sharif and Mohammed (Sharif and Mohammed, 2020) examined bintuitionistic fuzzy normed spaces, offering several key properties based on earlier contributions from (Jasim & Mohammad, 2017; Saadati & Park, 2006). The groundwork for 2-normed and general n-normed linear spaces was initially laid by S. Gähler (Gähler, 1964, 1969). Building upon this, Narayan and Vijayabalaji (Narayan and Vijayabalaji, 2005) expanded the theory into fuzzy n-normed spaces, drawing influence from Gähler's ideas (Gähler, 1969) and the work of Katsaras (Katsaras, 1984). The formulation of intuitionistic fuzzy n-normed linear spaces was later undertaken by Vijayabalaji et al. (Vijayabalaji et al., 2007), who also proved several foundational results. Independently, Branciari (Branciari, 2000) introduced the concept of rectangular metric spaces in 2000. Following this direction, Muteer and Mohammed (Muteer and Mohammed, 2023) proposed the notion of intuitionistic fuzzy rectangular b-normed spaces. Most recently, Badr and Mohammed (Badr and Mohammed, 2024) presented the idea of fuzzy rectangular n-normed spaces and analyzed some of their significant features.

Neutrosophic normed spaces, grounded in Smarandache's neutrosophic logic (Smarandache, 2005), have been widely studied for their ability to represent uncertainty via truth, indeterminacy, and falsity components. (Khan and Khan, 2022) extended classical normed space concepts into the neutrosophic setting using continuous t-norms and t-conorms, while Jenifer et al. (Jenifer et al., 2025) investigated statistical convergence in such spaces. Recent studies have expanded convergence theory within neutrosophic normed structures. Ahmad, Savas, and Mursaleen introduced deferred I-statistical rough convergence for difference sequences in neutrosophic normed spaces (Ahmad et al., 2026). Hossain and Mohiuddine further developed generalized difference I-convergence in neutrosophic n-normed spaces (Hossain & Mohiuddine, 2025). Moreover, Hossain, Mohiuddine, and Granados examined I-convergence in neutrosophic 2-normed spaces, establishing refined criteria for neutrosophic multi-normed settings (Hossain et al., 2025).

Separately, Mursaleen et al. (Mursaleen et al., 2009, 2010) explored convergence and sequence behavior in intuitionistic and fuzzy normed spaces, laying groundwork for structural generalizations. Inspired by these developments, Ahmad and Mursaleen (Ahmad & Mursaleen, 2025) studied on deferred statistical summability in neutrosophic n-normed linear space, Zarzour and Mohammed (Zarzour & Mohammed, 2025) the Cartesian product structure of intuitionistic fuzzy rectangular n-normed spaces and investigated its topological and algebraic properties.

A central motivation of this work is the need for a more robust framework that captures uncertainty in higher-dimensional normed structures. Classical and intuitionistic fuzzy n-normed spaces are often insufficient to model systems involving truth, indeterminacy, and falsity simultaneously. To address this gap, we develop the notion of neutrosophic rectangular n-normed spaces (NR-n-NS) and investigate their structural properties, including stability under Cartesian products and completeness and establish several supporting theorems.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}$ denotes the field of real numbers. In this section, we review some fundamental ideas and preliminaries regarding fuzzy rectangular n-normed space.

Definition 2.1. (Badr & Mohammad, 2024) *Let $\mathcal{U}$ be a vector space of dimension $d \geq n, n \in \mathbb{N}$ (natural numbers). A rectangular n-norm on $\mathcal{U}$ is a function $\|.,\dots,.\|$ on $\mathcal{U} \times \mathcal{U} \times \cdots \times \mathcal{U} = \mathcal{U}^n$ satisfying the following for $\tau_1, \tau_2, \dots, \tau_n, \mathfrak{f}, z \in \mathcal{U}$.*

(1). $\|\tau_1, \tau_2, \dots, \tau_n\| = 0 \Leftrightarrow \tau_1, \tau_2, \dots, \tau_n$ *are linearly dependent,*

(2). $\|\tau_1, \tau_2, \dots, \tau_n\|$ *is invariant under any permutation,*

(3). $\|\lambda\tau_1, \lambda\tau_2, \dots, \lambda\tau_n\| = |\lambda|\|\tau_1, \tau_2, \dots, \tau_n\|$ *for any* $\lambda \in \mathbb{R}$,

(4). $\|\tau_1, \tau_2, \dots, \tau_n + \mathfrak{f} + z\| \leq \|\tau_1, \tau_2, \dots, \tau_n\| + \|\tau_1, \tau_2, \dots, \mathfrak{f}\| + \|\tau_1, \tau_2, \dots, z\|$.

$\|.,\dots,.\|$ *is said to be a rectangular n-norm on $\mathcal{U}$ and the pair $(\mathcal{U}, \|.,\dots,.\|)$ is said to be a rectangular n-normed space.*

Definition 2.2. (Schweizer & Sklar, 1960) *A continuous t-norm ♠ is a binary operation on the interval* $[0,1]$, *that satisfies the following axioms:*

(1). *For each $e^* \in [0,1]$ implies that $e^* ♠ 1 = e^*$;*

(2). *♠ is associative and commutative;*

(3). *♠ is continuous,*

(4). *For each $e^*, s^*, z^*, d^* \in [0,1]$ and $e^* \leq z^*$ and $s^* \leq d^*$ implies that $e^* ♠ s^* \leq z^* ♠ d^*$.*

Definition 2.3. (Schweizer & Sklar, 1960) *A continuous t-conorm $\oslash$ is a binary operation on the interval* $[0,1]$ *which satisfies the following axioms:*

(1). *For each $e^* \in [0,1]$ implies that $e^* \oslash 0 = e^*$;*

(2). *$\oslash$ is associative and commutative;*

(3). *$\oslash$ is continuous;*

(4). *For each $e^*, s^*, z^*, d^* \in [0,1]$ and $\mathrm{e}^* \leq \mathrm{z}^*$ and $s^* \leq d^*$ implies that $e^* \oslash s^* \leq z^* \oslash d^*$.*

Definition 2.4. (Zarzour & Mohammad, 2025) *Let $\mathcal{U}$ be a vector space, ♠ be a continuous t-norm, $\oslash$ be a continuous t-conorm, a function $\Omega, M: \mathcal{U}^n \times (0, \infty) \to [0, \infty]$ is called intuitionistic fuzzy rectangular n-norm if it satisfies the following for all $(\tau_1, \tau_2, \dots, \tau_n, \mathfrak{f}, z) \in \mathcal{U}$ and $\varsigma, f, \grave{a} > 0$:*

(1). $\Omega(\tau_1, \tau_2, \dots, \tau_n, \varsigma) + M(\tau_1, \tau_2, \dots, \tau_n, \varsigma) \leq 1$,

(2). $\Omega(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = 0$, *for all $\varsigma \in \mathbb{R}$ with $\varsigma \leq 0$,*

(3). $\Omega(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = 1 \Leftrightarrow \tau_1, \tau_2, \dots, \tau_n$ *are linearly dependent,*

(4). $\Omega(\tau_1, \tau_2, \dots, \tau_n, \varsigma)$ *is invariant under any permutation of* $\tau_1, \tau_2, \dots, \tau_n$,

(5). $\Omega(\lambda\tau_1, \lambda\tau_2, \dots, \lambda\tau_n, \varsigma) = \Omega\left(\tau_1, \tau_2, \dots, \tau_n, \frac{\varsigma}{|\lambda|}\right)$, *if* $\lambda \in \mathbb{R} \setminus 0$,

(6). $\Omega(\tau_1, \tau_2, \dots, \tau_n + \mathfrak{f} + z, \varsigma + f + \grave{a}) \geq \Omega(\tau_1, \tau_2, \dots, \tau_n, \varsigma) ♠ \Omega(\tau_1, \tau_2, \dots, \mathfrak{f}, f) ♠ \Omega(\tau_1, \tau_2, \dots, z, \grave{a})$, (7). $\Omega(\tau_1,\tau_2,\dots,\tau_n,\varsigma)$ *is a non-decreasing function of* $\varsigma \in \mathbb{R}$ *and*

$$\lim_{\varsigma\to\infty}\Omega(\tau_1,\tau_2,\dots,\tau_n,\varsigma)=1,$$

(8). $M(\tau_1,\tau_2,\dots,\tau_n,\varsigma)=1,$

(9). $M(\tau_1,\tau_2,\dots,\tau_n,\varsigma)=0 \Leftrightarrow \tau_1,\tau_2,\dots,\tau_n$ *are linearly dependent,*

(10). $M(\tau_1,\tau_2,\dots,\tau_n,\varsigma)$ *is invariant under any permutation of* $\tau_1,\tau_2,\dots,\tau_n$,

(11). $M(\lambda\tau_1,\lambda\tau_2,\dots,\lambda\tau_n,\varsigma)=M\left(\tau_1,\tau_2,\dots,\tau_n,\frac{\varsigma}{|\lambda|}\right)$, *if* $\lambda\in\mathbb{R}\setminus 0$,

(12). $M(\tau_1,\tau_2,\dots,\tau_n+z,\varsigma+f+\grave{a})\le M(\tau_1,\tau_2,\dots,\tau_n,\varsigma)\oslash M(\tau_1,\tau_2,\dots,\text{ʄ},f)\oslash M(\tau_1,\tau_2,\dots,z,\grave{a})$,

(13). $M(\tau_1,\tau_2,\dots,\tau_n,\varsigma)$ *is a non-increasing function of* $\varsigma\in\mathbb{R}$ *and* $lim_{\varsigma\to\infty}M(\tau_1,\tau_2,\dots,\tau_n,\varsigma)=0.$

Hence, $(\mathcal{U},\Omega,M,\spadesuit,\oslash)$ *is called an intuitionistic fuzzy rectangular n-normed space (for short, IFR-n-NS).*

Example 2.5. *Let* $(\mathcal{U},\|\cdot,\dots,\cdot\|)$ *be a rectangular n-normed space. Define* $e^*\spadesuit s^*=e^*\cdot s^*$ *and* $e^*\oslash s^*=\min(1,e^*+s^*)$ *for each* $e^*,s^*\in[0,1]$.

Define as follows:

$$\Omega(\tau_1,\tau_2,\dots,\tau_n,\varsigma)=exp\left(-\frac{\|\tau_1,\tau_2,\dots,\tau_n\|}{\varsigma}\right),$$

$$M(\tau_1,\tau_2,\dots,\tau_n,\varsigma)=1-exp\left(-\frac{\|\tau_1,\tau_2,\dots,\tau_n\|}{\varsigma}\right),$$

where $\varsigma>0$ *and* $(\tau_1,\tau_2,\dots,\tau_n)\in\mathcal{U}$. *So* $(\mathcal{U},\Omega,M,\spadesuit,\oslash)$ *is an IFR-n-NS. Hence* $(X,\Omega,M,\spadesuit,\oslash)$ *is said to be a standard intuitionistic fuzzy rectangular n-normed space (for short, St-IFR-n-NS) induced by a rectangular n-normed space* $(\mathcal{U},\|\cdot,\dots,\cdot\|)$.

Definition 2.6. (Zarzour & Mohammad, 2025) *Let* $(\mathcal{U},\Omega,M,\spadesuit,\oslash)$ *be an IFR-n-NS. Then:*

(i). *A sequence* $\{\tau_n\}$ *in X is said to be convergent to* τ, *if for each* $\Upsilon\in(0,1)$ *and* $\varsigma>0$ *there is* $n_0\in N$ *in which*

$$\Omega(\tau_1,\tau_2,\dots,\tau_{n-1},\tau_n-\tau,\varsigma)>l-\Upsilon \text{ and } M(\tau_1,\tau_2,\dots,\tau_{n-1},\tau_n-\tau,\varsigma)<\Upsilon, \text{for all } n\ge n_0.$$

Or equivalently,

$$\lim_{\varsigma\to\infty}\Omega(\tau_1,\tau_2,\dots,\tau_{n-1},\tau_n-\tau,\varsigma)=1 \text{ and } \lim_{\varsigma\to\infty}M(\tau_1,\tau_2,\dots,\tau_{n-1},\tau_n-\tau,\varsigma)=0.$$

(ii). *A sequence* $\{\tau_n\}$ *in X is said to be Cauchy if, for all each* $\Upsilon\in(0,1)$ *and* $\varsigma>0$ *there is* $n_0\in N$ *in which*

$$\Omega(\tau_1,\tau_2,\dots,\tau_{n-1},\tau_n-\tau_K,\varsigma)>l-\Upsilon \text{ and } M(\tau_1,\tau_2,\dots,\tau_{n-1},\tau_n-\tau_\kappa,\varsigma)<\Upsilon, \text{for all } n,\kappa\ge n_0.$$

Or equivalently,

$$\lim_{\varsigma\to\infty} \Omega(\tau_1, \tau_2, \dots, \tau_{n-1}, \tau_n - \tau_K, \varsigma) = 1 \text{ and } \lim_{\varsigma\to\infty} M(\tau_1, \tau_2, \dots, \tau_{n-1}, \tau_n - \tau_K, \varsigma) = 0.$$

(iii). *An IFR-n-NS$(\mathcal{U}, \Omega, H)$ is said to be complete if, every Cauchy sequence converges.*

3. Cartesian product of neutrosophic rectangular n-normed spaces

In this section, we introduce the concept of a neutrosophic rectangular n-normed space and define the Cartesian product of two such spaces. We also establish and prove several related results.

Definition 3.1. *Let $\mathcal{U}$ be a vector space, ♠ be a continuous t-norm, $\oslash$ be a continuous t-conorm, a function $\Omega, M: \mathcal{U}^n \times (0,\infty) \to [0,\infty]$ is called neutrosophic rectangular n-norm if it satisfying the following for all $(\tau_1, \tau_2, \dots, \tau_n, \mathfrak{f}, z) \in \mathcal{U}$ and $\varsigma, f, \epsilon > 0$,*

(1). $\Omega(\tau_1, \tau_2, \dots, \tau_n, \varsigma) + M(\tau_1, \tau_2, \dots, \tau_n, \varsigma) \leq 1$,

(2). $\Omega(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = 0$, *for all* $\varsigma \in \mathbb{R}$ *with* $\varsigma \leq 0$,

(3). $\Omega(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = 1 \Leftrightarrow \tau_1, \tau_2, \dots, \tau_n$ *are linearly dependent,*

(4). $\Omega(\tau_1, \tau_2, \dots, \tau_n, \varsigma)$ *is invariant under any permutation of* $\tau_1, \tau_2, \dots, \tau_n$,

(5). $\Omega(\lambda\tau_1, \lambda\tau_2, \dots, \lambda\tau_n, \varsigma) = \Omega\left(\tau_1, \tau_2, \dots, \tau_n, \frac{\varsigma}{|\lambda|}\right)$, *if* $\lambda \in \mathbb{R} \setminus \{0\}$,

(6). $\Omega(\tau_1, \tau_2, \dots, \tau_n + \mathfrak{f} + z, \varsigma + f + \grave{a}) \geq \Omega(\tau_1, \tau_2, \dots, \tau_n, \varsigma) ♠ \Omega(\tau_1, \tau_2, \dots, \mathfrak{f}, f) ♠ \Omega(\tau_1, \tau_2, \dots, z, \grave{a})$,

(7). $\Omega(\tau_1, \tau_2, \dots, \tau_n, \varsigma)$ *is a non-decreasing function of* $\varsigma \in \mathbb{R}$ *and*

$$\lim_{\varsigma\to\infty} \Omega(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = 1,$$

(8). $M(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = 1$,

(9). $M(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = 0 \Leftrightarrow \tau_1, \tau_2, \dots, \tau_n$ *are linearly dependent,*

(10). $M(\tau_1, \tau_2, \dots, \tau_n, \varsigma)$ *is invariant under any permutation of* $\tau_1, \tau_2, \dots, \tau_n$,

(11). $M(\lambda\tau_1, \lambda\tau_2, \dots, \lambda\tau_n, \varsigma) = M\left(\tau_1, \tau_2, \dots, \tau_n, \frac{\varsigma}{|\lambda|}\right)$, *if* $\lambda \in \mathbb{R} \setminus \{0\}$,

(12). $M(\tau_1, \tau_2, \dots, \tau_n + z, \varsigma + f + \grave{a}) \leq M(\tau_1, \tau_2, \dots, \tau_n, \varsigma) \oslash M(\tau_1, \tau_2, \dots, \mathfrak{f}, f) \oslash M(\tau_1, \tau_2, \dots, z, \grave{a})$,

(13). $M(\tau_1, \tau_2, \dots, \tau_n, \varsigma)$ *is a non- increasing function of* $\varsigma \in \mathbb{R}$ *and*

$$\lim_{\varsigma\to\infty} M(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = 0.$$

(14). $L(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = 1$,

(15). $L(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = 0 \Leftrightarrow \tau_1, \tau_2, \dots, \tau_n$ *are linearly dependent,*

(16). $L(\tau_1, \tau_2, \dots, \tau_n, \varsigma)$ *is invariant under any permutation of* $\tau_1, \tau_2, \dots, \tau_n$,

(17). $L(\lambda\tau_1, \lambda\tau_2, \dots, \lambda\tau_n, \varsigma) = L\left(\tau_1, \tau_2, \dots, \tau_n, \frac{\varsigma}{|\lambda|}\right)$, *if* $\lambda \in \mathbb{R} \setminus \{0\}$,

(18). $L(\tau_1, \tau_2, \dots, \tau_n + z, \varsigma + f + \grave{a}) \leq L(\tau_1, \tau_2, \dots, \tau_n, \varsigma) \oslash L(\tau_1, \tau_2, \dots, \mathfrak{f}, f) \oslash L(\tau_1, \tau_2, \dots, z, \grave{a})$, (19). *$L(\tau_1, \tau_2, \dots, \tau_n, \varsigma)$ is a non- increasing function of $\varsigma \in \mathbb{R}$ and*

$$\lim_{\varsigma \to \infty} L(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = 0.$$

Hence, $(\mathcal{U}, \Omega, M, L, ♠, \oslash)$ is called a neutrosophic rectangular n-normed space (for short, NR- n-NS).

Example 3.2. *Let $(\mathcal{U}, \| \cdot, \dots, \cdot \|)$ be a rectangular n-normed space. Define the continuous t-norm ♠ and t-conorm $\oslash$ on $[0,1]$ by*

$$e^* ♠ s^* = \min\{e^*, s^*\}, e^* \oslash s^* = \min\{1, e^* + s^*\},$$

for each $e^*, s^* \in [0,1]$. Define the functions $\Omega, \mathrm{M}, \mathrm{L}: \mathcal{U}^n \times (0, \infty) \to [0,1]$ as follows:

$$\Omega(\tau_1, \dots, \tau_n, \varsigma) = \exp\left(-\frac{\|\tau_1, \dots, \tau_n\|}{\varsigma}\right),$$

$$\mathrm{M}(\tau_1, \dots, \tau_n, \varsigma) = 1 - \exp\left(-\frac{\|\tau_1, \dots, \tau_n\|}{\varsigma}\right),$$

$$\mathrm{L}(\tau_1, \dots, \tau_n, \varsigma) = 1 - \exp\left(-\frac{\|\tau_1, \dots, \tau_n\|}{\varsigma}\right),$$

where $\varsigma > 0$ and $(\tau_1, \dots, \tau_n) \in \mathcal{U}^n$. Then $(\mathcal{U}, \Omega, \mathrm{M}, \mathrm{L}, ♠, \oslash)$ is a neutrosophic rectangular n-normed space (NR-n-NS). This example uses the exponential decay form to model the neutrosophic components, where the norm controls the degree of membership, indeterminacy, and non-membership.

Note: In Definition 3.1, the mappings

$$\Omega, M: \mathcal{U}^n \times (0, \infty) \longrightarrow [0, \infty]$$

are allowed to take values in the extended non–negative real interval $[0, \infty]$. This choice is intentional and provides flexibility when dealing with limiting arguments in neutrosophic analysis, where indeterminacy or falsity components may, in principle, grow without an imposed upper bound.

However, the behaviour of Ω and M is fully controlled by the axioms of Definition 3.1. In particular,

(1). $\Omega(\cdot, \varsigma) \in [0,1]$ for all $\varsigma > 0$ and Ω is non-decreasing in ς;

(2). $M(\cdot, \varsigma) \in [0,1]$ for all $\varsigma > 0$ and M is non-increasing in ς;

(3). $\Omega + M \leq 1$.

Thus, although the codomain is formally written as $[0, \infty]$, the axioms *force* all admissible values of Ω and M to lie inside the bounded interval $[0,1]$. Consequently, expressions such as

$$M\left(\tau_1, \ldots, \tau_n, \frac{1}{2}\right) = \infty$$

cannot occur, since they violate conditions (1), (8), and (10) of the definition.

The inclusion of ∞ in the codomain simply allows the use of extended real analysis when handling limits (e.g., considering $\varsigma \to \infty$), but the axioms ensure that every neutrosophic rectangular n-norm actually attains values only in $[0,1]$.

This guarantees that the structure remains compatible with neutrosophic logic and preserves the bounded behaviour of truth, indeterminacy, and falsity degrees.

Definition 3.3. *Let $(\mathcal{U}, \Omega, \mathrm{M}, L, \spadesuit, \oslash)$ be an NR-n-NS. Then:*

(i). *A sequence $\{\tau_n\}$ in X is said to be convergent to τ, if for each $\Upsilon \in (0,1)$ and $\varsigma > 0$ there is $n_0 \in N$ in which*

$$\Omega(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau_n - \tau, \varsigma) > l - \Upsilon \text{ and } M(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau_n - \tau, \varsigma) < \Upsilon, \text{ and}$$
$$L(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau_n - \tau, \varsigma) < \Upsilon, \forall n \geq n_0.$$

Or equivalently,

$$\lim_{\varsigma \to \infty} \Omega(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau_n - \tau, \varsigma) = 1 \text{ and } \lim_{\varsigma \to \infty} M(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau_n - \tau, \varsigma) = 0, \text{ and}$$
$$\lim_{\varsigma \to \infty} L(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau_n - \tau, \varsigma) = 0.$$

(ii). *A sequence $\{\tau_n\}$ in X is said to be Cauchy if, for all each $\Upsilon \in (0,1)$ and $\varsigma > 0$ there is $n_0 \in N$ in which*

$$\Omega(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau_n - \tau_K, \varsigma) > l - \Upsilon \text{ and } M(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau_n - \tau_\kappa, \varsigma) < \Upsilon \text{ and}$$
$$L(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau_n - \tau_\kappa, \varsigma) < \Upsilon, \text{for all } n, \kappa \geq n_0.$$

Or equivalently,

$$\lim_{\varsigma \to \infty} \Omega(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau_n - \tau_K, \varsigma) = 1 \text{ and } \lim_{\varsigma \to \infty} M(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau_n - \tau_K, \varsigma) = 0 \text{ and}$$
$$\lim_{\varsigma \to \infty} L(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau_n - \tau_K, \varsigma) = 0.$$

(iii). *An NR-n-NS $(\mathcal{U}, \Omega, M, L)$ is said to be complete if, every Cauchy sequence converges.*

Definition 3.4. *Let $(\mathcal{U}, \Omega_1, M_1, L_1, \spadesuit, \oslash)$ and $(\mathcal{U}, \Omega_1, M_1, L_2, \spadesuit, \oslash)$ be two NR-n-NS. The Cartesian product of $(\mathcal{U}, \Omega_1, M_1, L_1, \spadesuit, \oslash)$ and $(\mathcal{H}, \Omega_2, M_2, L_2, \spadesuit, \oslash)$ is the product space $(\mathcal{U} \times \mathcal{H}, \Omega, M, L, \spadesuit, \oslash)$, where $\mathcal{U} \times \mathcal{H}$ is the Cartesian product of the sets $\mathcal{U}^n \times \mathcal{H}^n$ and Ω, M are a function*

$\Omega: \left((\mathcal{U}^n \times \mathcal{H}^n) \times (0, \infty) \to [0,1]\right)$, $M: \left((\mathcal{U}^n \times \mathcal{H}^n) \times (0, \infty) \to [0,1]\right)$ *and* $L: \left((\mathcal{U}^n \times \mathcal{H}^n) \times (0, \infty) \to [0,1]\right)$ *are given by:*

$\Omega: (\tau_1, \tau_2, \ldots, \tau_n, \mu_1, \mu_2, \ldots, \mu_n), \varsigma) = \Omega_1(\tau_1, \tau_2, \ldots, \tau_n, \varsigma) \spadesuit \Omega_2(\mu_1, \mu_2, \ldots, \mu_n, \varsigma)$ *and*

$M: (\tau_1, \tau_2, \ldots, \tau_n, \mu_1, \mu_2, \ldots, \mu_n), \varsigma) = M_1(\tau_1, \tau_2, \ldots, \tau_n, \varsigma) \oslash M_2(\mu_1, \mu_2, \ldots, \mu_n, \varsigma)$,

$L: (\tau_1, \tau_2, \dots, \tau_n, \mu_1, \mu_2, \dots, \mu_n), \varsigma) = L_1(\tau_1, \tau_2, \dots, \tau_n, \varsigma) \oslash L_2(\mu_1, \mu_2, \dots, \mu_n, \varsigma).$

for all $(\tau_1, \tau_2, \dots, \tau_n, \mu_1, \mu_2, \dots, \mu_n) \in \mathcal{U}^n \times \mathcal{H}^n$ *and* $\varsigma > 0$.

Example 3.5. *Let* $n \in \mathbb{N}$, $n \geq 1$, *and let* $\mathcal{U} = \mathbb{R}$, $\mathcal{H} = \mathbb{R}$. *Consider two NR-*n*-neutrosophic normed spaces* $(\mathcal{U}, \Omega_1, M_1, L_1, \spadesuit, \oslash)$ *and* $(\mathcal{H}, \Omega_2, M_2, L_2, \spadesuit, \oslash)$. *For* $\boldsymbol{\tau} = (\tau_1, \tau_2, \dots, \tau_n) \in \mathcal{U}^n$ *and* $\varsigma > 0$, *define*

$$\Omega_1(\boldsymbol{\tau}, \varsigma) = \frac{\varsigma}{\varsigma + \sum_{i=1}^{n} |\tau_i|}, M_1(\boldsymbol{\tau}, \varsigma) = \frac{\sum_{i=1}^{n-1} |\tau_i|}{\varsigma + \sum_{i=1}^{n} |\tau_i|}, L_1(\boldsymbol{\tau}, \varsigma) = \frac{|\tau_n|}{\varsigma + \sum_{i=1}^{n} |\tau_i|}.$$

For $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_n) \in \mathcal{H}^n$ *and* $\varsigma > 0$, *define*

$$\Omega_2(\boldsymbol{\mu}, \varsigma) = exp\left(-\frac{\sum_{i=1}^{n} |\mu_i|}{\varsigma}\right), M_2(\boldsymbol{\mu}, \varsigma) = \frac{\sum_{i=1}^{n-1} |\mu_{i+1} - \mu_i|}{1 + \sum_{i=1}^{n} |\mu_i|}, L_2(\boldsymbol{\mu}, \varsigma) = 1 - \Omega_2(\boldsymbol{\mu}, \varsigma).$$

Choose the neutrosophic combination operations by

$$e^* \spadesuit s^* = \min\{e^*, s^*\}, e^* \oslash s^* = \max\{e^*, s^*\}, e^*, s^* \in [0,1].$$

The Cartesian product space is

$$(\mathcal{U} \times \mathcal{H}, \Omega, M, L, \spadesuit, \oslash),$$

where for each $(\boldsymbol{\tau}, \boldsymbol{\mu}) \in \mathcal{U}^n \times \mathcal{H}^n$ *and* $\varsigma > 0$ *we set*

$$\Omega(\boldsymbol{\tau}, \boldsymbol{\mu}, \varsigma) = \Omega_1(\boldsymbol{\tau}, \varsigma) \spadesuit \Omega_2(\boldsymbol{\mu}, \varsigma) = \min\{\Omega_1(\boldsymbol{\tau}, \varsigma), \Omega_2(\boldsymbol{\mu}, \varsigma)\},$$

$$M(\boldsymbol{\tau}, \boldsymbol{\mu}, \varsigma) = M_1(\boldsymbol{\tau}, \varsigma) \oslash M_2(\boldsymbol{\mu}, \varsigma) = \max\{M_1(\boldsymbol{\tau}, \varsigma), M_2(\boldsymbol{\mu}, \varsigma)\},$$

$$L(\boldsymbol{\tau}, \boldsymbol{\mu}, \varsigma) = L_1(\boldsymbol{\tau}, \varsigma) \oslash L_2(\boldsymbol{\mu}, \varsigma) = \max\{L_1(\boldsymbol{\tau}, \varsigma), L_2(\boldsymbol{\mu}, \varsigma)\}.$$

Therefore, each of Ω, M, L *maps*

$$(\mathcal{U}^n \times \mathcal{H}^n) \times (0, \infty) \to [0,1],$$

since $\Omega_1, \Omega_2, M_1, M_2, L_1, L_2$ *take values in* $[0,1]$ *and* $\min, \max$ *preserve this range. The chosen formulas are standard and satisfy the usual monotonicity and normalization conditions required for NR-*n*-neutrosophic normed spaces; hence the Cartesian product indeed defines an NR-*n*-neutrosophic normed space.*

Next we show that if $\mathcal{U}$ and $\mathcal{Y}$ are NR-n-NS, then their Cartesian product will also be an NR-n-NS.

Theorem 3.6. *Let* $(\mathcal{U}, \Omega_1, M_1, L_1, \spadesuit, \oslash)$ *and* $(\mathcal{U}, \Omega_2, M_2, L_2, \spadesuit, \oslash)$ *be an NR-*n*-NSs. Then* $(\mathcal{U}^n \times \mathcal{H}^n, \Omega, M, L, \spadesuit, \oslash)$ *is an NR-*n*-NS.*

Proof. Given $(\mathcal{U}, \Omega_1, M_1, L_1, \spadesuit, \oslash)$ and $(\mathcal{U}, \Omega_2, M_2, L_2, \spadesuit, \oslash)$ are NR-n-NSs.

(1) Since $\Omega_1(\tau_1, \tau_2, \dots, \tau_n, \varsigma) + M_1(\mu_1, \mu_2, \dots, \mu_n, \varsigma) \leq 1$ and $\Omega_2(\tau_1, \tau_2, \dots, \tau_n, \varsigma) + M_2(\mu_1, \mu_2, \dots, \mu_n, \varsigma) \leq 1$ $\Rightarrow \Omega\big((\tau_1, \tau_2, \dots, \tau_n, \mu_1, \mu_2, \dots, \mu_n), \varsigma\big) + M\big((\tau_1, \tau_2, \dots, \tau_n, \mu_1, \mu_2, \dots, \mu_n), \varsigma\big) \leq 1.$

(2) Since $\Omega_1(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = 0$ and $\Omega_2(\mu_1, \mu_2, \dots, \mu_n, \varsigma) = 0$, for all $\varsigma > 0$

$\Rightarrow \Omega\big((\tau_1, \tau_2, \dots, \tau_n, \mu_1, \mu_2, \dots, \mu_n), \varsigma\big) = 0.$

(3) Since $\Omega_1(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = 1 \Leftrightarrow \tau_1, \tau_2, \dots, \tau_n$ are linearly dependent and $\Omega_2(\mu_1, \mu_2, \dots, \mu_n, \varsigma) = 1 \Leftrightarrow \mu_1, \mu_2, \dots, \mu_n$ are linearly dependent $\Rightarrow \Omega\big((\tau_1, \tau_2, \dots, \tau_n, \mu_1, \mu_2, \dots, \mu_n), \varsigma\big) = 1 \Leftrightarrow (\tau_1, \tau_2, \dots, \tau_n, \mu_1, \mu_2, \dots, \mu_n)$ are linearly dependent.

(4) Since $\Omega_1(\lambda\tau_1, \lambda\tau_2, \dots, \lambda\tau_n, \varsigma) = \Omega_1\left(\tau_1, \tau_2, \dots, \tau_n, \frac{\varsigma}{|\lambda|}\right)$ and $\Omega_2(\lambda\mu_1, \lambda\mu_2, \dots, \lambda\mu_n, \varsigma) = \Omega_2\left(\mu_1, \mu_2, \dots, \mu_n, \frac{\varsigma}{|\lambda|}\right)$

$\Rightarrow \Omega(\lambda(\tau_1, \tau_2, \dots, \tau_n, \mu_1, \mu_2, \dots, \mu_n), \varsigma)$

$= \Omega_1(\lambda\tau_1, \lambda\tau_2, \dots, \lambda\tau_n, \varsigma) \spadesuit \Omega_2(\lambda\mu_1, \lambda\mu_2, \dots, \lambda\mu_n, \varsigma)$

$= \Omega_1\left(\tau_1, \tau_2, \dots, \tau_n, \frac{\varsigma}{|\lambda|}\right) \spadesuit \Omega_2(\mu_1, \mu_2, \dots, \mu_n), \frac{\varsigma}{|\lambda|})$

$= \Omega\left((\tau_1, \tau_2, \dots, \tau_n, \mu_1, \mu_2, \dots, \mu_n), \frac{\varsigma}{|\lambda|}\right).$

(5) Since $\Omega_1(\tau_1, \tau_2, \dots, \tau_n + \text{ƒ} + z, \varsigma + f + \grave{a}) \geq \Omega_1(\tau_1, \tau_2, \dots, \tau_n, \varsigma) \spadesuit \Omega_1(\tau_1, \tau_2, \dots, \text{ƒ}, f) \spadesuit \Omega_1(\tau_1, \tau_2, \dots, z, c)$ and $\Omega_2(\mu_1, \mu_2, \dots, \mu_n + J + w, \varsigma + \mathbb{R} + c) \geq \Omega_2(\mu_1, \mu_2, \dots, \mu_n, \varsigma) \spadesuit \Omega_2(\mu_1, \mu_2, \dots, J, f) \spadesuit \Omega_2(\mu_1, \mu_2, \dots, w, c)$

$\Rightarrow \Omega\Big((\tau_1, \tau_2, \dots, \tau_n, \mu_1, \mu_2, \dots, \mu_n) + (\tau_1, \tau_2, \dots, f_1, \mu_1, \mu_2, \dots, \mathcal{J}) + (\tau_1, \tau_2, \dots, z, \mu_1, \mu_2, \dots, w), (\varsigma + f + \grave{a})\Big)$

$\Rightarrow \Omega\big(\tau_1, \tau_2, \dots, \tau_n + \text{ƒ} + z, \mu_1, \mu_2, \dots, \mu_n + J + w, (\varsigma + \dagger + \grave{a})\big)$

$= \Omega_1(\tau_1, \tau_2, \dots, \tau_n + \text{ƒ} + z, \varsigma + f + \grave{a}) \spadesuit \Omega_2(\mu_1, \mu_2, \dots, \mu_n + \mathcal{J} + w, \varsigma + f + \grave{a})$

$\geq \Omega_1(\tau_1, \tau_2, \dots, \tau_n, \varsigma) \spadesuit \Omega_1(\tau_1, \tau_2, \dots, \text{f. f}) \spadesuit \Omega_1(\tau_1, \tau_2, \dots, z, \grave{a}) \spadesuit \Omega_2(\mu_1, \mu_2, \dots, \mu_n, \varsigma) \spadesuit \Omega_2(\mu_1, \mu_2, \dots, J,$
$\dagger) \spadesuit \Omega_2(\mu_1, \mu_2, \dots, w, c)$

$\geq \Omega_1(\tau_1, \tau_2, \dots, \tau_n, \varsigma) \spadesuit \Omega_2(\mu_1, \mu_2, \dots, \mu_n, \varsigma) \spadesuit \Omega_1(\tau_1, \tau_2, \dots, \text{ƒ}, f) \spadesuit \Omega_2(\mu_1, \mu_2, \dots, J, \mathbb{R}) \spadesuit \Omega_1(\aleph_1, \aleph_2, \dots, z, c)$
$\spadesuit \Omega_2(\mu_1, \mu_2, \dots, w, c)$

$= \Omega\big((\tau_1, \tau_2, \dots, \tau_n, \mu_1, \mu_2, \dots, \mu_n), \varsigma\big) \spadesuit \Omega\big((\tau_1, \tau_2, \dots, h', \mu_1, \mu_2, \dots, \delta), f\big) \spadesuit \Omega\big((\tau_1, \tau_2, \dots, z, \mu_1, \mu_2, \dots, w), c\big).$

(6) Since $\Omega_1(\tau_1, \tau_2, \dots, \tau_n, \varsigma): (0, \infty) \to [0,1]$ is continuous in ς and $\Omega_2(\mu_1, \mu_2, \dots, \mu_n, \varsigma): (0, \infty) \to [0,1]$ is continuous in $\varsigma \Rightarrow \Omega\big((\tau_1, \tau_2, \dots, \tau_n, \mu_1, \mu_2, \dots, \mu_n), \varsigma\big): (0, \infty) \to [0,1]$ is continuous in ς.

(7) Since $\lim_{\varsigma\to\infty} \Omega_1(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = 1$ and $\lim_{\varsigma\to\infty} \Omega_2(\mu_1, \mu_2, \dots, \mu_n, \varsigma) = 1$

$\Rightarrow \lim_{\varsigma\to\infty} \Omega\big((\tau_1, \tau_2, \dots, \tau_n, \mu_1, \mu_2, \dots, \mu_n), \varsigma\big) = 1.$

(8) Since $M_1(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = 1$ and $M_2(\mu_1, \mu_2, \dots, \mu_n, \varsigma) = 1$, for all $\varsigma > 0$

$\Rightarrow M\big((\tau_1, \tau_2, \dots, \tau_n, \mu_1, \mu_2, \dots, \mu_n), \varsigma\big) = 1.$

(9) Since $M_1(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = 0 \Leftrightarrow \tau_1, \tau_2, \dots, \tau_n$ are linearly dependent and $M_2(\mu_1, \mu_2, \dots, \mu_n, \varsigma) = 0 \Leftrightarrow$
$\mu_1, \mu_2, \dots, \mu_n$ are linearly dependent

$\Rightarrow M\big((\tau_1, \tau_2, \dots, \tau_n, \mu_1, \mu_2, \dots, \mu_n), \varsigma\big) = 0$

$\Leftrightarrow (\tau_1, \tau_2, \dots, \tau_n, \mu_1, \mu_2, \dots, \mu_n)$ are linearly dependent.

(10) Since $M_1(\lambda\tau_1,\lambda\tau_2,\ldots,\lambda\tau_n,\varsigma) = M_1\left(\tau_1,\tau_2,\ldots,\tau_n,\frac{\varsigma}{|\lambda|}\right)$ and $M_2(\lambda\mu_1,\lambda\mu_2,\ldots,\lambda\mu_n,\varsigma) = M_2\left(\mu_1,\mu_2,\ldots,\mu_n,\frac{\varsigma}{|\lambda|}\right)$

$\Rightarrow M(\lambda(\tau_1,\tau_2,\ldots,\tau_n,\mu_1,\mu_2,\ldots,\mu_n),\varsigma)$

$= M_1(\lambda\tau_1,\lambda\tau_2,\ldots,\lambda\tau_n,\varsigma) \oslash M_2(\lambda\mu_1,\lambda\mu_2,\ldots,\lambda\mu_n,\varsigma)$

$= M_1\left(\tau_1,\tau_2,\ldots,\tau_n,\frac{\varsigma}{|\lambda|}\right) \oslash M_2(\mu_1,\mu_2,\ldots,\mu_n),\frac{\varsigma}{|\lambda|})$

$= M\left((\tau_1,\tau_2,\ldots,\tau_n,\mu_1,\mu_2,\ldots,\mu_n),\frac{\varsigma}{|\lambda|}\right).$

Since $M_1(\tau_1,\tau_2,\ldots,\tau_n+\zeta+z,\varsigma+f+\grave{a}) \le M_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma) \oslash M_1(\tau_1,\tau_2,\ldots,\mathfrak{f},f) \oslash M_1(\tau_1,\tau_2,\ldots,z,\mathrm{c})$ and $M_2(\mu_1,\mu_2,\ldots,\mu_n+\mathrm{J}+\mathrm{w},\varsigma+f+\grave{a})$

$\le M_2(\mu_1,\mu_2,\ldots,\mu_n,\varsigma) \oslash M_2(\mu_1,\mu_2,\ldots,\mathrm{J},\uparrow)\spadesuit\Omega_2(\mu_1,\mu_2,\ldots,\mathrm{w},\grave{a})$

$\Rightarrow M\big((\tau_1,\tau_2,\ldots,\tau_n,\mu_1,\mu_2,\ldots,\mu_n)+(\tau_1,\tau_2,\ldots,f_1,\mu_1,\mu_2,\ldots,\delta)+(\tau_1,\tau_2,\ldots,\mathrm{z},\mu_1,\mu_2,\ldots,\mathrm{w}),(\varsigma+f+\grave{a})\big)$

$\Rightarrow M(\tau_1,\tau_2,\ldots,\tau_n+\mathrm{h}+\mathrm{z},\mu_1,\mu_2,\ldots,\mu_n+\mathrm{J}+\mathrm{w},(\varsigma+f+\grave{a}))$

$= M_1(\tau_1,\tau_2,\ldots,\tau_n+\mathrm{z},\varsigma+f+\grave{a}) \oslash M_2(\mu_1,\mu_2,\ldots,\mu_n+\mathrm{J}+\mathrm{w},\varsigma+f+\grave{a})$

$\le M_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma) \oslash M_1(\tau_1,\tau_2,\ldots,\mathfrak{f},f) \oslash M_1(\tau_1,\tau_2,\ldots,z,\grave{a}) \oslash M_2(\mu_1,\mu_2,\ldots,\mu_n,\varsigma) \oslash$
$M_2(\mu_1,\mu_2,\ldots,\mathrm{J},f) \oslash M_2(\mu_1,\mu_2,\ldots,\mathrm{w},\mathrm{c})$

$\le M_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma) \oslash M_2(\mu_1,\mu_2,\ldots,\mu_n,\varsigma) \oslash M_1(\tau_1,\tau_2,\ldots,\mathfrak{f},f) \oslash M_2(\mu_1,\mu_2,\ldots,\mathrm{J},\mathbb{R}) \oslash M_1(\aleph_1,\aleph_2,\ldots,\mathrm{z},\mathrm{c})$
$\oslash M_2(\mu_1,\mu_2,\ldots,\mathrm{w},\mathrm{c})$

$= M\big((\tau_1,\tau_2,\ldots,\tau_n,\mu_1,\mu_2,\ldots,\mu_n),\varsigma\big) \vee M\big((\tau_1,\tau_2,\ldots,\gamma_1,\mu_1,\mu_2,\ldots,\delta),f\big) \oslash M\big((\tau_1,\tau_2,\ldots,\mathrm{z},\mu_1,\mu_2,\ldots,\mathrm{w}),\mathrm{c}\big).$

Since $M_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma):(0,\infty)\to[0,1]$ is continuous in ς and $M_2(\mu_1,\mu_2,\ldots,\mu_n,\varsigma):(0,\infty)\to[0,1]$ is continuous in $\varsigma \Rightarrow M\big((\tau_1,\tau_2,\ldots,\tau_n,\mu_1,\mu_2,\ldots,\mu_n),\varsigma\big):(0,\infty)\to[0,1]$ is continuous in ς.

(13) Since $\lim_{\varsigma\to\infty} M_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma) = 0$ and $\lim_{\varsigma\to\infty} M_2(\mu_1,\mu_2,\ldots,\mu_n,\varsigma) = 0$

$\Rightarrow \lim_{\varsigma\to\infty} M\big((\tau_1,\tau_2,\ldots,\tau_n,\mu_1,\mu_2,\ldots,\mu_n),\varsigma\big) = 0.$

(14) Since $L_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma) = 1$ and $L_2(\mu_1,\mu_2,\ldots,\mu_n,\varsigma) = 1$, for all $\varsigma > 0$

$\Rightarrow L\big((\tau_1,\tau_2,\ldots,\tau_n,\mu_1,\mu_2,\ldots,\mu_n),\varsigma\big) = 1.$

(15) Since $L_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma) = 0 \Leftrightarrow \tau_1,\tau_2,\ldots,\tau_n$ are linearly dependent and $L_2(\mu_1,\mu_2,\ldots,\mu_n,\varsigma) = 0 \Leftrightarrow \mu_1,\mu_2,\ldots,\mu_n$ are linearly dependent

$= L_1\left(\tau_1,\tau_2,\ldots,\tau_n,\frac{\varsigma}{|\lambda|}\right) \oslash L_2(\mu_1,\mu_2,\ldots,\mu_n),\frac{\varsigma}{|\lambda|})$

$= L\left((\tau_1,\tau_2,\ldots,\tau_n,\mu_1,\mu_2,\ldots,\mu_n),\frac{\varsigma}{|\lambda|}\right).$

Since $L_1(\tau_1,\tau_2,\ldots,\tau_n+\zeta+z,\varsigma+f+\grave{a}) \le L_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma) \oslash L_1(\tau_1,\tau_2,\ldots,\mathfrak{f},f) \oslash L_1(\tau_1,\tau_2,\ldots,z,\ \mathrm{c}\ )$ and $L_2(\mu_1,\mu_2,\ldots,\mu_n+\mathrm{J}+\mathrm{w},\varsigma+f+\grave{a})$

$\leq \mathrm{L}_2(\mu_1, \mu_2, \dots, \mu_\mathrm{n}, \varsigma) \oslash \mathrm{L}_2(\mu_1, \mu_2, \dots, \mathrm{J}, \uparrow)$♠$\Omega_2(\mu_1, \mu_2, \dots, \mathrm{w}, \grave{a})$

$\Rightarrow \mathrm{L}\big((\tau_1, \tau_2, \dots, \tau_\mathrm{n}, \mu_1, \mu_2, \dots, \mu_\mathrm{n}) + (\tau_1, \tau_2, \dots, f_1, \mu_1, \mu_2, \dots, \delta) + (\tau_1, \tau_2, \dots, \mathrm{z}, \mu_1, \mu_2, \dots, \mathrm{w}), (\varsigma + f + \grave{a})\big)$

$\Rightarrow \mathrm{L}(\tau_1, \tau_2, \dots, \tau_\mathrm{n} + \mathrm{h} + \mathrm{z}, \mu_1, \mu_2, \dots, \mu_\mathrm{n} + \mathrm{J} + \mathrm{w}, (\varsigma + f + \grave{a}))$

$= \mathrm{L}_1(\tau_1, \tau_2, \dots, \tau_\mathrm{n} + \mathrm{z}, \varsigma + f + \grave{a}) \oslash \mathrm{L}_2(\mu_1, \mu_2, \dots, \mu_\mathrm{n} + \mathrm{J} + \mathrm{w}, \varsigma + f + \grave{a})$

$\leq \mathrm{L}_1(\tau_1, \tau_2, \dots, \tau_n, \varsigma) \oslash \mathrm{L}_1(\tau_1, \tau_2, \dots, \mathfrak{f}, f) \oslash \mathrm{L}_1(\tau_1, \tau_2, \dots, z, \grave{a}) \oslash \mathrm{L}_2(\mu_1, \mu_2, \dots, \mu_\mathrm{n}, \varsigma) \oslash \mathrm{L}_2(\mu_1, \mu_2, \dots, \mathrm{J}, f) \oslash \mathrm{L}_2(\mu_1, \mu_2, \dots, \mathrm{w}, \mathrm{c})$

$\leq \mathrm{L}_1(\tau_1, \tau_2, \dots, \tau_\mathrm{n}, \varsigma) \oslash \mathrm{L}_2(\mu_1, \mu_2, \dots, \mu_\mathrm{n}, \varsigma) \oslash \mathrm{L}_1(\tau_1, \tau_2, \dots, \mathfrak{f}, f) \oslash \mathrm{L}_2(\mu_1, \mu_2, \dots, \mathrm{J}, \mathbb{R}) \oslash \mathrm{L}_1(\aleph_1, \aleph_2, \dots, \mathrm{z}, \mathrm{c}) \oslash \mathrm{L}_2(\mu_1, \mu_2, \dots, \mathrm{w}, \mathrm{c})$

$= \mathrm{L}\big((\tau_1, \tau_2, \dots, \tau_\mathrm{n}, \mu_1, \mu_2, \dots, \mu_\mathrm{n}), \varsigma\big) \vee \mathrm{L}\big((\tau_1, \tau_2, \dots, \gamma_1, \mu_1, \mu_2, \dots, \delta), f\big) \oslash \mathrm{L}\big((\tau_1, \tau_2, \dots, \mathrm{z}, \mu_1, \mu_2, \dots, \mathrm{w}), \mathrm{c}\big)$.

Since $\mathrm{L}_1(\tau_1, \tau_2, \dots, \tau_\mathrm{n}, \varsigma): (0, \infty) \to [0,1]$ is continuous in ς and $\mathrm{L}_2(\mu_1, \mu_2, \dots, \mu_\mathrm{n}, \varsigma): (0, \infty) \to [0,1]$ is continuous in ς

$\Rightarrow \mathrm{L}\big((\tau_1, \tau_2, \dots, \tau_\mathrm{n}, \mu_1, \mu_2, \dots, \mu_\mathrm{n}), \varsigma\big): (0, \infty) \to [0,1]$ is continuous in ς.

(17) Since $\lim_{\varsigma\to\infty} \mathrm{L}_1(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = 0$ and $\lim_{\varsigma\to\infty} \mathrm{L}_2(\mu_1, \mu_2, \dots, \mu_n, \varsigma) = 0$

$\Rightarrow \lim_{\varsigma\to\infty} \mathrm{L}\big((\tau_1, \tau_2, \dots, \tau_\mathrm{n}, \mu_1, \mu_2, \dots, \mu_\mathrm{n}), \varsigma\big) = 0$.

This completes the proof.

Subsequently, we demonstrate that the converse of Theorem 3.6 is valid.

Theorem 3.7. *If* $(\mathcal{U}^\mathrm{n} \times \mathcal{H}^\mathrm{n}, \Omega, \mathrm{M}, L,$ ♠$, \oslash)$ *is an NR-n-NS, then* $(\mathcal{U}, \Omega_1, \mathrm{M}_1, \mathrm{L}_1,$ ♠$, \oslash)$ *and* $(\mathcal{U}, \Omega_2, \mathrm{M}_2, \mathrm{L}_2,$ ♠$, \oslash)$ *are also NR-n-NSs by defining*

$\Omega_1(\tau_1, \tau_2, \dots, \tau_\mathrm{n}, \varsigma) = \Omega\big((\tau_1, \tau_2, \dots, \tau_\mathrm{n}, 0), \varsigma\big)$ *and*

$\mathrm{M}_1(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = \mathrm{M}\big((\tau_1, \tau_2, \dots, \tau_n, 0), \varsigma\big)$,

$\mathrm{L}_1(\tau_1, \tau_2, \dots, \tau_n, \varsigma) = \mathrm{L}\big((\tau_1, \tau_2, \dots, \tau_n, 0), \varsigma\big)$,

$\Omega_2(\mu_1, \mu_2, \dots, \mu_n, \varsigma) = \Omega\big((0, \mu_1, \mu_2, \dots, \mu_n), \varsigma\big)$ *and*

$\mathrm{M}_2(\mu_1, \mu_2, \dots, \mu_n, \varsigma) = \mathrm{M}\big((0, \mu_1, \mu_2, \dots, \mu_n), \varsigma\big)$

$\mathrm{L}_2(\mu_1, \mu_2, \dots, \mu_n, \varsigma) = \mathrm{L}\big((0, \mu_1, \mu_2, \dots, \mu_n), \varsigma\big)$

for all $\tau_1, \tau_2, \dots, \tau_\mathrm{n} \in \mathcal{U}$ *and* $\mu_1, \mu_2, \dots, \mu_\mathrm{n} \in \mathcal{H}$ *and* $\varsigma > 0$.

Proof. (1) $\Omega_1(\tau_1, \tau_2, \dots, \tau_\mathrm{n}, \varsigma) + \mathrm{M}_1(\tau_1, \tau_2, \dots, \tau_\mathrm{n}, \varsigma)$

$= \Omega\big((\tau_1, \tau_2, \dots, \tau_\mathrm{n}, 0), \varsigma\big) + \mathrm{M}\big((\tau_1, \tau_2, \dots, \tau_\mathrm{n}, 0), \varsigma\big) \leq 1$

$\Rightarrow \Omega_1(\tau_1, \tau_2, \dots, \tau_\mathrm{n}, \varsigma) + \mathrm{M}_1(\tau_1, \tau_2, \dots, \tau_\mathrm{n}, \varsigma) \leq 1$.

(2) $\Omega_1(\tau_1, \tau_2, \dots, \tau_\mathrm{n}, \varsigma) = \Omega\big((\tau_1, \tau_2, \dots, \tau_\mathrm{n}, 0), \varsigma\big) = 0$ for all $\tau_1, \tau_2, \dots, \tau_\mathrm{n} \in \mathcal{U}$

$\Rightarrow \Omega_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma) = 0$ and $M_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma) = M\big((\tau_1,\tau_2,\ldots,\tau_n,0),\varsigma\big) = 1$ and $L_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma) = L\big((\tau_1,\tau_2,\ldots,\tau_n,0),\varsigma\big) = 1$

For all $\tau_1,\tau_2,\ldots,\tau_n \in \mathcal{U} \Rightarrow M_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma) = 1$.

(3) For all $\varsigma > 0, 1 = \Omega_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma) = \Omega\big((\tau_1,\tau_2,\ldots,\tau_n,0),\varsigma\big)$

$\Leftrightarrow \tau_1,\tau_2,\ldots,\tau_n$ are linearly dependent and $0 = M_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma) = M\big((\tau_1,\tau_2,\ldots,\tau_n,0),\varsigma\big)$ and $0 = L_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma) = L\big((\tau_1,\tau_2,\ldots,\tau_n,0),\varsigma\big)$

$\Leftrightarrow \tau_1,\tau_2,\ldots,\tau_n$ are linearly dependent.

(4) For all $\varsigma > 0$,

$\Omega_1(\lambda\tau_1,\lambda\tau_2,\ldots,\lambda\tau_n,\varsigma) = \Omega(\lambda(\tau_1,\tau_2,\ldots,\tau_n,0),\varsigma)$

$\Omega\left((\tau_1,\tau_2,\ldots,\tau_n,0),\frac{\varsigma}{|\lambda|}\right) = \Omega_1\left(\tau_1,\tau_2,\ldots,\tau_n,\frac{\varsigma}{|\lambda|}\right)$ for all $\lambda \in \mathbb{R}\setminus\{0\}$ and

$M_1(\lambda\tau_1,\lambda\tau_2,\ldots,\lambda\tau_n,\varsigma) = M(\lambda(\tau_1,\tau_2,\ldots,\tau_n,0),\varsigma)$

$M\left((\tau_1,\tau_2,\ldots,\tau_n,0),\frac{\varsigma}{|\lambda|}\right) = M_1\left(\tau_1,\tau_2,\ldots,\tau_n,\frac{\varsigma}{|\lambda|}\right)$ for all $\lambda \in \mathbb{R}\setminus\{0\}$.and

$L_1(\lambda\tau_1,\lambda\tau_2,\ldots,\lambda\tau_n,\varsigma) = L(\lambda(\tau_1,\tau_2,\ldots,\tau_n,0),\varsigma)$

$L\left((\tau_1,\tau_2,\ldots,\tau_n,0),\frac{\varsigma}{|\lambda|}\right) = L_1\left(\tau_1,\tau_2,\ldots,\tau_n,\frac{\varsigma}{|\lambda|}\right)$ for all $\lambda \in \mathbb{R}\setminus\{0\}$.

(5) For all $\tau_1,\tau_2,\ldots,\tau_n + \oint + z \in \mathcal{U}$ and $\varsigma_1,\varsigma_2,\varsigma_3 > 0$. Then $\Omega_1\big(\tau_1,\tau_2,\ldots,\tau_n + \mathfrak{h} + z,(\varsigma_1+\varsigma_2+\varsigma_3)\big)$

$= \Omega\big((\tau_1,\tau_2,\ldots,\tau_n + \zeta + z,0),(\varsigma_1+\varsigma_2+\varsigma_3)\big)$

$= \Omega\big((\tau_1,\tau_2,\ldots,\tau_n,0) + (\tau_1,\tau_2,\ldots,f,0) + (\tau_1,\tau_2,\ldots,z,0),(\varsigma_1+\varsigma_2+\varsigma_3)\big)$

$\geq \Omega\big((\tau_1,\tau_2,\ldots,\tau_n,0),\varsigma_1\big)\spadesuit\Omega\big((\tau_1,\tau_2,\ldots,f,0),\varsigma_2\big)\spadesuit\Omega\big((\tau_1,\tau_2,\ldots,z,0),\varsigma_3\big)$

$\geq \Omega_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma_1)\spadesuit\Omega_1(\tau_1,\tau_2,\ldots,\zeta,\varsigma_2)\spadesuit\Omega_1(\tau_1,\tau_2,\ldots,z,\varsigma_3)$

$\Omega_1\big(\tau_1,\tau_2,\ldots,\tau_n + \mathfrak{h} + z,(\varsigma_1+\varsigma_2+\varsigma_3)\big)$

$\geq \Omega_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma_1)\spadesuit\Omega_1(\tau_1,\tau_2,\ldots,f,\varsigma_2)\spadesuit\Omega_1(\tau_1,\tau_2,\ldots,z,\varsigma_3)$ and

$M_1\big(\tau_1,\tau_2,\ldots,\tau_n + \mathfrak{h} + z,(\varsigma_1+\varsigma_2+\varsigma_3)\big)$

$= M\big((\tau_1,\tau_2,\ldots,\tau_n + \mathfrak{h} + z,0),(\varsigma_1+\varsigma_2+\varsigma_3)\big)$

$= M\big((\tau_1,\tau_2,\ldots,\tau_n,0) + (\tau_1,\tau_2,\ldots,f,0) + (\tau_1,\tau_2,\ldots,z,0),(\varsigma_1+\varsigma_2+\varsigma_3)\big)$

$\leq M\big((\tau_1,\tau_2,\ldots,\tau_n,0),\varsigma_1\big) \oslash M\big((\tau_1,\tau_2,\ldots,f,0),\varsigma_2\big) \oslash M\big((\tau_1,\tau_2,\ldots,z,0),\varsigma_3\big)$

$\leq M_1(\tau_1,\tau_2,\ldots,\tau_n,\varsigma_1) \oslash M_1(\tau_1,\tau_2,\ldots,\zeta_2) \oslash M_1(\tau_1,\tau_2,\ldots,z,\varsigma_3)$

$M_1\big(\tau_1,\tau_2,\ldots,\tau_n + \mathfrak{h} + z,(\varsigma_1+\varsigma_2+\varsigma_3)\big)$

$\leq \mathrm{M}_1(\tau_1,\tau_2,\dots,\tau_\mathrm{n},\varsigma_1) \oslash \mathrm{M}_1(\tau_1,\tau_2,\dots,\mathfrak{h},\varsigma_2) \oslash \mathrm{M}_1(\tau_1,\tau_2,\dots,\mathrm{z},\varsigma_3)$ and $\mathrm{L}_1\big(\tau_1,\tau_2,\dots,\tau_\mathrm{n}+\mathfrak{h}+\mathrm{z},(\varsigma_1+\varsigma_2+\varsigma_3)\big)$

$= \mathrm{L}\big((\tau_1,\tau_2,\dots,\tau_\mathrm{n}+\mathfrak{h}+\mathrm{z},0),(\varsigma_1+\varsigma_2+\varsigma_3)\big)$

$= \mathrm{L}\big((\tau_1,\tau_2,\dots,\tau_n,0)+(\tau_1,\tau_2,\dots,\mathrm{f},0)+(\tau_1,\tau_2,\dots,\mathrm{z},0),(\varsigma_1+\varsigma_2+\varsigma_3)\big)$

$\leq \mathrm{L}\big((\tau_1,\tau_2,\dots,\tau_n,0),\varsigma_1\big) \oslash \mathrm{L}\big((\tau_1,\tau_2,\dots,f,0),\varsigma_2\big) \oslash \mathrm{L}\big((\tau_1,\tau_2,\dots,z,0),\varsigma_3\big)$

$\leq \mathrm{L}_1(\tau_1,\tau_2,\dots,\tau_\mathrm{n},\varsigma_1) \oslash \mathrm{L}_1(\tau_1,\tau_2,\dots,\zeta_2) \oslash \mathrm{L}_1(\tau_1,\tau_2,\dots,\mathrm{z},\varsigma_3)$

$\mathrm{L}_1\big(\tau_1,\tau_2,\dots,\tau_\mathrm{n}+\mathfrak{h}+\mathrm{z},(\varsigma_1+\varsigma_2+\varsigma_3)\big) \leq \mathrm{L}_1(\tau_1,\tau_2,\dots,\tau_\mathrm{n},\varsigma_1) \oslash \mathrm{L}_1(\tau_1,\tau_2,\dots,\mathfrak{h},\varsigma_2) \oslash \mathrm{L}_1(\tau_1,\tau_2,\dots,\mathrm{z},\varsigma_3)$.

(6) $\Omega_1(\tau_1,\tau_2,\dots,\tau_\mathrm{n},\varsigma) = \Omega\big((\tau_1,\tau_2,\dots,\tau_\mathrm{n},0)\varsigma\big)$ is a continuous in ς and $\mathrm{M}_1\left((\tau_1,\tau_2,\dots,\tau_n,\varsigma) = \mathrm{M}\big((\tau_1,\tau_2,\dots,\tau_n,0),\varsigma\big)\right.$ is a continuous in ς and $\mathrm{L}_1\left((\tau_1,\tau_2,\dots,\tau_n,\varsigma) = \mathrm{L}\big((\tau_1,\tau_2,\dots,\tau_n,0),\varsigma\big)\right.$ is a continuous in ς.

(7) $\lim_{\varsigma\to\infty}\Omega_1(\tau_1,\tau_2,\dots,\tau_\mathrm{n},\varsigma) = \lim_{\varsigma\to\infty}\Omega\big((\tau_1,\tau_2,\dots,\tau_\mathrm{n},0),\varsigma\big) = 1$ and $\lim_{\varsigma\to\infty}\mathrm{M}_1(\tau_1,\tau_2,\dots,\tau_\mathrm{n},\varsigma) = \lim_{\varsigma\to\infty}\mathrm{M}\big((\tau_1,\tau_2,\dots,\tau_\mathrm{n},0),\varsigma\big) = 0$ and $\lim_{\varsigma\to\infty}\mathrm{L}_1(\tau_1,\tau_2,\dots,\tau_\mathrm{n},\varsigma) = \lim_{\varsigma\to\infty}\mathrm{L}\big((\tau_1,\tau_2,\dots,\tau_\mathrm{n},0),\varsigma\big) = 0$. Then $(\mathcal{U},\Omega_1,\mathrm{M}_1,\mathrm{L}_1,\spadesuit,\oslash)$ is an NR-n-NS.

Similarly, we can prove that $(\mathcal{U},\Omega_2,\mathrm{M}_2,\mathrm{L}_2,\spadesuit,\oslash)$ is a NR-n-NS.

The following theorem establishes that if sequences in $\mathcal{U}$ and $\mathcal{H}$ are convergent, then their Cartesian product also converges.

Theorem 3.8. *Let τ_n be a sequence in an NR-n-NS $(\mathcal{U},\Omega_1,\mathrm{M}_1,\mathrm{L}_1,\spadesuit,\oslash)$ converging to τ in $\mathcal{U}$, μ_n be a sequence in an NR-n-NSs $(U,\Omega_2,\mathrm{M}_2,\mathrm{L}_2,\spadesuit,\oslash)$ converging to μ in $\mathcal{H}$, then (τ_n,μ_n) is a sequence in an NR-n-NS $(\mathcal{U}\times\mathcal{H},\Omega,\mathrm{M},L,\spadesuit,\oslash)$ converge to $(\tau,\mu)\in\mathcal{U}\times\mathcal{H}$.*

Proof. Let $\Upsilon\in(0,1)$ and $\varsigma>0$. Since $\{\tau_\mathrm{n}\}$ is a convergence sequence in $\mathcal{U}$, there is $\mathrm{n}_1\in\mathrm{N}$ in which $\Omega_1(\tau_1,\tau_2,\dots,\tau_{\mathrm{n}-1},\tau_\mathrm{n}-\tau,\varsigma) > 1-\Upsilon$ and $\mathrm{M}_1(\tau_1,\tau_2,\dots,\tau_{\mathrm{n}-1},\tau_\mathrm{n}-\tau,\varsigma) < \Upsilon$ and $\mathrm{L}_1(\tau_1,\tau_2,\dots,\tau_{\mathrm{n}-1},\tau_\mathrm{n}-\tau,\varsigma) < \Upsilon$, for all $\mathrm{n}\geq\mathrm{n}_1$.

Since $\{\mu_n\}$ is a convergence sequence in $\mathcal{H}$, there is $\mathrm{n}_2\in\mathrm{N}$ in which $\Omega_2(\mu_1,\mu_2,\dots,\mu_\mathrm{n}-\mu,\varsigma) > 1-\Upsilon$ and $\mathrm{M}_2(\mu_1,\mu_2,\dots,\mu_\mathrm{n}-\mu,\varsigma) < \Upsilon$ and $\mathrm{L}_2(\mu_1,\mu_2,\dots,\mu_\mathrm{n}-\mu,\varsigma) < \Upsilon$, for all $\mathrm{n}\geq\mathrm{n}_2$.

Then, for all $\Upsilon\in(0,1)$ and $\varsigma>0$, there is $\mathrm{n}_0\in\mathrm{N}$, where $\mathrm{n}_0=\max\{\mathrm{n}_1,\mathrm{n}_2\}$ in which

$\Omega(\tau_1,\tau_2,\dots,\tau_{\mathrm{n}-1},\mu_1,\mu_2,\dots,\mu_{\mathrm{n}-1},(\tau_\mathrm{n},\mu_\mathrm{n})-(\tau,\mu),\varsigma)$

$\geq \Omega_1(\tau_1,\tau_2,\dots,\tau_{\mathrm{n}-1},\tau_\mathrm{n}-\tau,\varsigma)\spadesuit\Omega_2(\mu_1,\mu_2,\dots,\mu_{\mathrm{n}-1},\mu_\mathrm{n}-\mu,\varsigma)$

$> (1-\Upsilon)\spadesuit(1-\Upsilon) > 1-\Upsilon$

and $\mathrm{M}(\tau_1,\tau_2,\dots,\tau_{\mathrm{n}-1},\mu_1,\mu_2,\dots,\mu_{\mathrm{n}-1},(\tau_\mathrm{n},\mu_\mathrm{n})-(\tau,\mu),\varsigma)$

$\leq \mathrm{M}_1(\tau_1, \tau_2, \dots, \tau_{n-1}, \tau_n - \tau, \varsigma) \oslash \mathrm{M}_2(\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n - \mu, \varsigma)$

$< \Upsilon \oslash \Upsilon < \Upsilon$

and $\mathrm{L}(\tau_1, \tau_2, \dots, \tau_{n-1}, \mu_1, \mu_2, \dots, \mu_{n-1}, (\tau_n, \mu_n) - (\tau, \mu), \varsigma)$

$\leq \mathrm{L}_1(\tau_1, \tau_2, \dots, \tau_{n-1}, \tau_n - \tau, \varsigma) \oslash \mathrm{L}_2(\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n - \mu, \varsigma)$

$< \Upsilon \oslash \Upsilon < \Upsilon$.

Thus, $\{(\tau_n, \mu_n)\}$ converges to (τ, μ).

The next result confirms the validity of the converse of Theorem 3.8.

Theorem 3.9. *Let (τ_n, μ_n) be a sequence in an NR-n-NS $(\mathcal{U} \times \mathcal{H}, \Omega, M, L, ♠, \oslash)$, then τ_n is a sequence in an NR-n-NS $(\mathcal{U}, \Omega_1, M_1, L_1, ♠, \oslash)$ converge to τ in $\mathcal{U}$ and $\{\mu_n\}$ be a sequence in an NR-n-NS$(\mathcal{H}, \Omega_2, M_2, L_2, ♠, \oslash)$ converge to μ in $\mathcal{H}$.*

Proof. The proof of this theorem follows directly and is therefore omitted.

The upcoming theorem proves that Cauchy sequences in $\mathcal{U}$ and $\mathcal{H}$ yield a Cauchy sequence in their Cartesian product.

Theorem 3.10. *Let $\{\tau_n\}$ be a Cauchy sequence in an NR-n-NS $(\mathcal{U}, \Omega_1, \mathrm{M}_1, \mathrm{L}_1, ♠, \oslash)$ and $\{\mu_n\}$ be a Cauchy sequence in an NR-n-NS $(U, \Omega_2, \mathrm{M}_2, \mathrm{L}_2, ♠, \oslash)$, then $\{(\tau_n, \mu_n)\}$ is a Cauchy sequence in an NR-n-NS $(\mathcal{U} \times \mathcal{H}, \Omega, \mathrm{M}, L, ♠, \oslash)$.*

Proof. By Theorem 3.6, $(\mathcal{U} \times \mathcal{H}, \Omega, \mathrm{M}, \mathrm{L}, ♠, \oslash)$ is an NR-n-NS. Since τ_n be a Cauchy sequence in an NR- n-NS $(\mathcal{U}, \Omega_1, \mathrm{M}_1, \mathrm{L}_1, ♠, \oslash)$, then for all $\Upsilon \in (0,1)$ and $\varsigma > 0$, there is $\mathrm{n}_1 \in \mathrm{N}$ in which $\Omega_1(\tau_1, \tau_2, \dots, \tau_n - \tau_\kappa, \varsigma) > 1 - \Upsilon$ and $\mathrm{M}_1(\tau_1, \tau_2, \dots, \tau_n - \tau_\kappa, \varsigma) < \Upsilon$, for all $\mathrm{n}, \kappa \geq \mathrm{n}_1$ and $\mathrm{L}_1(\tau_1, \tau_2, \dots, \tau_n - \tau_\kappa, \varsigma) < \Upsilon$, for all $\mathrm{n}, \kappa \geq \mathrm{n}_1$.

Since $\{\mu_n\}$ be a Cauchy sequence in an NR- n-NS $(\mathcal{H}, \Omega_2, \mathrm{M}_2, \mathrm{L}_2, ♠, \oslash)$, then for all $\Upsilon \in (0,1)$ and $\varsigma > 0$, there is $\mathrm{n}_2 \in \mathrm{N}$ in which $\Omega_2(\mu_1, \mu_2, \dots, \mu_n - \mu_K, \varsigma) > 1 - \Upsilon$ and $\mathrm{M}_2(\mu_1, \mu_2, \dots, \mu_n - \mu_K, \varsigma) < \Upsilon$, for all $\mathrm{n}, \kappa \geq \mathrm{n}_2$ and $\mathrm{L}_2(\mu_1, \mu_2, \dots, \mu_n - \mu_K, \varsigma) < \Upsilon$, for all $\mathrm{n}, \kappa \geq \mathrm{n}_2$. Then for all $\Upsilon \in (0,1)$ and $\varsigma > 0$, there is $\mathrm{n}_0 \in \mathrm{N}$ where, $\mathrm{n}_0 = \max\{\mathrm{n}_1, \mathrm{n}_2\}$, for all $\mathrm{n}, \kappa \geq \mathrm{n}_0$.

$\Omega(\tau_1, \tau_2, \dots, \tau_{n-1}, \mu_1, \mu_2, \dots, \mu_{n-1}, (\tau_n, \mu_n) - (\tau_K, \mu_K), \varsigma)$

$\geq \Omega_1(\tau_1, \tau_2, \dots, \tau_{n-1}, \tau_n - \tau_K, \varsigma) ♠ \Omega_2(\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n - \mu_K, \varsigma)$

$> (1 - \Upsilon) ♠ (1 - \Upsilon) > 1 - \Upsilon$ and

$\mathrm{M}(\tau_1, \tau_2, \dots, \tau_{n-1}, \mu_1, \mu_2, \dots, \mu_{n-1}, (\tau_n, \mu_n) - (\tau_K, \mu_K), \varsigma)$

$\leq \mathrm{M}_1(\tau_1, \tau_2, \dots, \tau_{n-1}, \tau_n - \tau_K, \varsigma) \oslash \mathrm{M}_2(\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n - \mu_K, \varsigma)$

$< \Upsilon \oslash \Upsilon < \Upsilon$ and

$\mathrm{L}(\tau_1, \tau_2, \dots, \tau_{n-1}, \mu_1, \mu_2, \dots, \mu_{n-1}, (\tau_n, \mu_n) - (\tau_K, \mu_K), \varsigma)$

$\leq \mathrm{L}_1(\tau_1, \tau_2, \dots, \tau_{n-1}, \tau_n - \tau_K, \varsigma) \oslash \mathrm{L}_2(\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n - \mu_K, \varsigma)$

$< \Upsilon \oslash \Upsilon < \Upsilon$.

Thus, $\{(\tau_n, \mu_n)\}$ is a Cauchy sequence in $(\mathcal{U} \times \mathcal{H}, \Omega, \mathrm{M}, L, ♠, \oslash)$.

The next theorem establishes that the converse of Theorem 3.10 also holds.

Theorem 3.11. *If $\{(\tau_n, \mu_n)\}$ is a Cauchy sequence in an NR-n-NS $(\mathcal{U} \times \mathcal{H}, \Omega, M, L, ♠, \oslash)$, then $\{\tau_n\}$ is a Cauchy sequence in an NR-n-NS $(\mathcal{U}, \Omega_1, M_1, L_1, ♠, \oslash)$ and $\{\mu_n\}$ is a Cauchy sequence in an NR-n-NS $(\mathcal{H}, \Omega_2, M_2, L_2, ♠, \oslash)$.*

Proof. Assume $\{(\tau_n, \mu_n)\}$ is Cauchy in $(\mathcal{U} \times \mathcal{H}, \Omega, M, L, ♠, \oslash)$. Then, for every $\Upsilon \in (0,1)$ and every $\varsigma > 0$ there exists $\mathrm{N} \in \mathbb{N}$ such that for all $n, \mathrm{K} \geq \mathrm{N}$, we have

$$\Omega(\tau_1, \tau_2, \dots, \tau_{n-1}, \mu_1, \mu_2, \dots, \mu_{n-1}, (\tau_n, \mu_n) - (\tau_K, \mu_K), \varsigma) > 1 - \Upsilon,$$

$$\mathrm{M}(\tau_1, \tau_2, \dots, \tau_{n-1}, \mu_1, \mu_2, \dots, \mu_{n-1}, (\tau_n, \mu_n) - (\tau_K, \mu_K), \varsigma) < \Upsilon,$$

$$\mathrm{L}(\tau_1, \tau_2, \dots, \tau_{n-1}, \mu_1, \mu_2, \dots, \mu_{n-1}, (\tau_n, \mu_n) - (\tau_K, \mu_K), \varsigma) < \Upsilon.$$

By the product-space definitions, we have, for each n, K and $\varsigma > 0$,

$$\Omega(\tau_1, \tau_2, \dots, \tau_{n-1}, \mu_1, \mu_2, \dots, \mu_{n-1}, (\tau_n, \mu_n) - (\tau_K, \mu_K), \varsigma) = \Omega_1(\tau_1, \tau_2, \dots, \tau_{n-1}, \tau_n - \tau_K, \varsigma) ♠ \Omega_2(\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n - \mu_K, \varsigma),$$

and

$$\mathrm{M}(\tau_1, \tau_2, \dots, \tau_{n-1}, \mu_1, \mu_2, \dots, \mu_{n-1}, (\tau_n, \mu_n) - (\tau_K, \mu_K), \varsigma) = \mathrm{M}_1(\tau_1, \tau_2, \dots, \tau_{n-1}, \tau_n - \tau_K, \varsigma) \oslash \mathrm{M}_2(\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n - \mu_K, \varsigma),$$

and

$$\mathrm{L}(\tau_1, \tau_2, \dots, \tau_{n-1}, \mu_1, \mu_2, \dots, \mu_{n-1}, (\tau_n, \mu_n) - (\tau_K, \mu_K), \varsigma) = \mathrm{L}_1(\tau_1, \tau_2, \dots, \tau_{n-1}, \tau_n - \tau_K, \varsigma) \oslash \mathrm{L}_2(\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n - \mu_K, \varsigma).$$

Using the assumed order-properties of ♠ and $\oslash$ we deduce for all $n, \mathrm{K} \geq \mathrm{N}$:

$$\Omega_1(\tau_1, \tau_2, \dots, \tau_{n-1}, \tau_n - \tau_K, \varsigma) ♠ \Omega_2(\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n - \mu_K, \varsigma) \leq \Omega_1(\tau_1, \tau_2, \dots, \tau_{n-1}, \tau_n - \tau_K, \varsigma),$$

so

$\Omega_1(\tau_1, \tau_2, \dots, \tau_{n-1}, \tau_n - \tau_K, \varsigma)$

$\geq \Omega(\tau_1, \tau_2, \dots, \tau_{n-1}, \mu_1, \mu_2, \dots, \mu_{n-1}, (\tau_n, \mu_n) - (\tau_K, \mu_K), \varsigma)$

$> 1 - \Upsilon$.

Similarly,

$\Omega_2(\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n - \mu_K, \varsigma)$

$\geq \Omega(\tau_1, \tau_2, \dots, \tau_{n-1}, \mu_1, \mu_2, \dots, \mu_{n-1}, (\tau_n, \mu_n) - (\tau_K, \mu_K), \varsigma)$

$> 1 - \Upsilon$.

For the indeterminacy and falsity parts, since $a \oslash b \geq a$ and $a \oslash b \geq b$, we obtain, for all $n, \mathrm{K} \geq \mathrm{N}$:

$\mathrm{M}_1(\tau_1, \tau_2, \dots, \tau_{\mathrm{n}-1}, \tau_{\mathrm{n}} - \tau_{\mathrm{K}}, \varsigma)$

$\leq \mathrm{M}_1(\tau_1, \tau_2, \dots, \tau_{\mathrm{n}-1}, \tau_{\mathrm{n}} - \tau_{\mathrm{K}}, \varsigma) \oslash \mathrm{M}_2(\mu_1, \mu_2, \dots, \mu_{\mathrm{n}-1}, \mu_{\mathrm{n}} - \mu_{\mathrm{K}}, \varsigma)$

$= \mathrm{M}(\tau_1, \tau_2, \dots, \tau_{\mathrm{n}-1}, \mu_1, \mu_2, \dots, \mu_{\mathrm{n}-1}, (\tau_{\mathrm{n}}, \mu_{\mathrm{n}}) - (\tau_{\mathrm{K}}, \mu_{\mathrm{K}}), \varsigma)$

$< \Upsilon$,

and

$\mathrm{M}_2(\mu_1, \mu_2, \dots, \mu_{\mathrm{n}-1}, \mu_{\mathrm{n}} - \mu_{\mathrm{K}}, \varsigma) < \Upsilon$.

Analogous inequalities hold for $\mathrm{L}_1, \mathrm{L}_2$:

$L_1(\mu_1, \mu_2, \dots, \mu_{\mathrm{n}-1}, \mu_{\mathrm{n}} - \mu_{\mathrm{K}}, \varsigma) < \Upsilon$ and $L_2(\mu_1, \mu_2, \dots, \mu_{\mathrm{n}-1}, \mu_{\mathrm{n}} - \mu_{\mathrm{K}}, \varsigma) < \Upsilon$.

Combining the above, we see that for every $\Upsilon \in (0,1)$ and $\varsigma > 0$ there exists N such that for all $n, \mathrm{K} \geq \mathrm{N}$, $\Omega_1(\tau_1, \tau_2, \dots, \tau_{\mathrm{n}-1}, \tau_{\mathrm{n}} - \tau_{\mathrm{K}}, \varsigma) > 1 - \Upsilon$, and $M_1(\tau_1, \tau_2, \dots, \tau_{\mathrm{n}-1}, \tau_{\mathrm{n}} - \tau_{\mathrm{K}}, \varsigma) < \Upsilon$, and $L_1(\tau_1, \tau_2, \dots, \tau_{\mathrm{n}-1}, \tau_{\mathrm{n}} - \tau_{\mathrm{K}}, \varsigma) < \Upsilon$, and the analogous three inequalities for Ω_2, M_2, L_2.

Thus $\{\tau_n\}$ is Cauchy in $(\mathcal{U}, \Omega_1, M_1, L_1, \spadesuit, \oslash)$ and $\{\mu_n\}$ is Cauchy in $(\mathcal{H}, \Omega_2, M_2, L_2, \spadesuit, \oslash)$, as required.

Theorem 3.12. *If* $(\mathcal{U}, \Omega_1, \mathrm{M}_1, \mathrm{L}_1, \spadesuit, \oslash)$ *and* $(\mathcal{U}, \Omega_2, \mathrm{M}_2, \mathrm{L}_2, \spadesuit, \oslash)$ *are complete an NR-n-NSs, then the product* $(\mathcal{U} \times \mathcal{H}, \Omega, \mathrm{M}, L, \spadesuit, \oslash)$ *is complete an NR-n-NS.*

Proof. Let $(\tau_{\mathrm{n}}, \mu_{\mathrm{n}})$ be a Cauchy sequence in $\mathcal{U} \times \mathcal{H}$. Then, Theorem 3.11

$\Rightarrow \{\tau_{\mathrm{n}}\}$ ia a Cauchy sequence in $(\mathcal{U}, \Omega_1, \mathrm{M}_1, \mathrm{L}_1, \spadesuit, \oslash)$ and $\{\mu_n\}$ is a Cauchy sequence in $(U, \Omega_2, \mathrm{M}_2, \mathrm{L}_2, \spadesuit, \oslash)$.

Since $\mathcal{U}$ and $\mathcal{H}$ are complete, therefore $\{\tau_{\mathrm{n}}\}$ is a convergence sequence in $\mathcal{U}$ and $\{\mu_{\mathrm{n}}\}$ is a convergence sequence in $\mathcal{H}$.

Now, Theorem 3.8 $\Rightarrow \{(\tau_{\mathrm{n}}, \mu_{\mathrm{n}})\}$ is a convergence sequence in $\mathcal{U} \times \mathcal{H}$.

The result below can be established using methods similar to those employed in Theorems 3.9 and 3.10.

Theorem 3.13. *If* $(\mathcal{U} \times \mathcal{H}, \Omega, \mathrm{M}, L, \spadesuit, \oslash)$ *be a complete an NR-n-NS, then* $(\mathcal{U}, \Omega_1, \mathrm{M}_1, \mathrm{L}_1, \spadesuit, \oslash)$ *and* $(\mathcal{U}, \Omega_2, \mathrm{M}_2, \mathrm{L}_2, \spadesuit, \oslash)$ *are complete an NR-n-NSs.*

Proof. Let $\{\tau_{\mathrm{n}}\}$ be a Cauchy sequence in $\mathcal{U}$, $\{\mu_{\mathrm{n}}\}$ be a Cauchy sequence in $\mathcal{H}$. Then, Theorem 3.10 $\Rightarrow (\tau_{\mathrm{n}}, \mu_{\mathrm{n}})$ is a Cauchy sequence in $\mathcal{U} \times \mathcal{H}$. Since $\mathcal{U} \times \mathcal{H}$ is complete $\Rightarrow \{(\tau_{\mathrm{n}}, \mu_{\mathrm{n}})\}$ is a convergence sequence in $\mathcal{U} \times \mathcal{H}$ by Theorem 3.9 $\Rightarrow \{\tau_{\mathrm{n}}\}$ is a convergence sequence in $\mathcal{U}$ and $\{\mu_{\mathrm{n}}\}$ is a convergence sequence in $\mathcal{H}$.

4. Conclusion

In this work, we introduced and systematically developed the framework of neutrosophic rectangular n-normed spaces (NR-n-NS), establishing their fundamental structure and analytical properties.

The study showed that the Cartesian product of two NR-n-NS naturally inherits the neutrosophic rectangular n-normed structure, thereby preserving the underlying neutrosophic behavior and geometric characteristics of the component spaces. Moreover, it was proved that the Cartesian product of complete NR-n-NS remains complete, ensuring the stability of convergence processes within the product environment. Several supporting results and theorems were obtained to strengthen the theoretical foundation of the proposed framework. These findings provide a robust platform for further exploration of neutrosophic functional analysis and open new avenues for applications involving uncertainty, indeterminacy, and multi-dimensional neutrosophic modeling.

5. Future directions

This research lays the groundwork for a variety of potential future studies. One direction for continued exploration involves extending the theory of neutrosophic rectangular n-normed spaces into operator theory particularly focusing on the definition and analysis of linear mappings and functionals in such frameworks. Another area of interest is constructing neutrosophic analogues of inner product spaces, which may yield valuable geometric insights.

Further investigation could address the topological characteristics and continuity-related aspects of Cartesian products in neutrosophic normed environments. Topics such as compactness, connectedness, and convergence behavior within these structures merit deeper analysis. Moreover, applying these theoretical developments to practical domains involving uncertainty-such as decision science, control mechanisms, and data-driven modeling-could lead to impactful applications.

Finally, identifying fixed point results, establishing criteria for completeness in more generalized neutrosophic settings, and incorporating probabilistic or statistical perspectives within the framework of rectangular n-normed spaces represent open and intriguing challenges for future research.

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