Dera Natung Government College Research Journal
Volume 10 Issue 1, pp. 79-90, January-December 2025

ISSN (Print): 2456-8228
ISSN (Online): 2583-5483

Research Article

Characterizations of summability methods derived by q-Cesàro matrix

Fadime Gökçe*

Faculty of Science, University of Pamukkale, Denizli, Turkey.

Cite as: Gökçe, F. (2025). Characterizations of summability methods derived by q-Cesàro matrix. Dera Natung Government College Research Journal, 10 (1), 79-90. https://doi.org/10.56405/dngcrj.2025.10.01.05

Received on: 24.09.2025,
Revised on: 14.11.2025
Accepted on: 01.12.2025,
Published on: 30.12.2025.

*Corresponding Author: **Fadime Gökçe** (fgokce@pau.edu.tr)

Abstract: The present study mainly aims to introduce the absolute summability method $|C^q,\varphi|_s$ derived from the transformation matrix obtained by the q-Cesàro matrix and to establish the necessary and sufficient conditions for $|C^q,\varphi|_s \Rightarrow |C^p,\delta|, |C^q,\varphi| \Rightarrow |C^p,\delta|_s$ and $|C^q,\varphi|_s \Rightarrow |C^p,\delta|_s$ where $1 \leq s < \infty$. In addition, by obtaining new results for certain specific situations, the importance of using the q-analogue concept in absolute summability theory is emphasized and the scope of previous studies is expanded.

Keywords: Absolute summability, q-Cesàro matrix, q-analogue, Summability methods.

MSC 2020: 40D25, 40F05

1. Introduction

The theory of summability and its subfields has been a longstanding area of interest for researchers in the engineering sciences, applied mathematics, and functional analysis. The exploration of summability methods, sequence spaces, and their associated transformations plays a fundamental role in the analysis of series convergence and its wide-ranging applications. This area of research frequently intersects with various mathematical disciplines, including calculus, approximation theory, quantum mechanics, probability theory, and Fourier analysis.

Over the years, summability theory has undergone substantial development—not only through the formulation of summability methods based on classical matrices such as Hölder, Fibonacci, Euler, Cesàro, Lucas, Hausdorff, and Nörlund, and the associated sequence spaces—but also through the study of matrix transformations and their rich topological and algebraic structures. In recent years, growing attention has been directed toward absolute summability methods and the novel sequence spaces they have generated, offering new perspectives and deepening the understanding within this dynamic field. Another branch of research related to statistical convergence has also attracted considerable interest from scientists. Consequently, contemporary studies in summability theory continue to make significant contributions to the advancement of mathematical literature (see (Ellidokuzoğlu et al., 2018; Gökçe, 2022, 2024; Gökçe & Sarıgöl, 2019, 2020a, 2020b; Gürdal & Yamancı, 2015; Indumathi et al., 2023; Yamancı & Gürdal, 2015; Huban & Gürdal, 2021; Kara & Bayrakdar, 2021; Kişi et al., 2025; Kişi & Gürdal, 2022; Kişi & Erhan, 2018; Savaş et al., 2022; Sarıgöl, 2010; Yaying & Kara, 2021)).

In this paper, firstly, the absolute summability method $|C^q,\varphi|_s$ derived by transformation matrix obtained by the q-Cesàro matrix is introduced and then, the necessary and sufficient conditions for $|C^q,\varphi|_s \Rightarrow |C^p,\delta|, |C^q,\varphi| \Rightarrow |C^p,\delta|_s$ and $|C^q,\varphi|_s \Rightarrow |C^p,\delta|_s$ are established, where $1 \leq s < \infty$. In addition, by obtaining new results for certain situations, the importance of using the q-analogue concept in absolute summability theory is emphasized and the scope of previous studies is expanded.

First, let us recall some basic notations and concepts.

By ω, l_∞, c and l_s $(s \geq 1)$, we represent the set of all sequences of complex entries, the sequence spaces of all bounded, convergent sequences and also the space of all s-absolutely convergent series, respectively. Also, throughout the paper, $\mathbb{N} = \{0,1,2,3,\ldots\}$. Let $\Lambda = (\lambda_{ji})$ be any infinite matrix of complex entries and U, V be two subspaces of ω. If the series

$$\Lambda_j(u) = \sum_{i=0}^{\infty} \lambda_{ji} u_i$$

converges for all $j \epsilon \mathbb{N}$, then, it is said that the Λ-transform of the sequence $u = (u_i)$ is defined by $\Lambda(u) = (\Lambda_j(u))$. Also, it is said that Λ determines a matrix transformation from the space U into another space V, and the class of all infinite matrices $\Lambda : U \to V$ is represented by (U, V). If $\lambda_{ji} = 0$ for $i > j$, and otherwise $\lambda_{ji} \neq 0$ for all j, i, then it is said that Λ is a triangle.

Unless otherwise specified, throughout this study, we assume that $\Lambda = (\lambda_{ji})$ is an infinite matrix of complex components for all $j, i \in \mathbb{N}$, (φ_j) and (δ_j) are any sequences of positive numbers, and also s^* indicates the conjugate of s that is $1/s + 1/s^* = 1$ for $s > 0$, $1/s^* = 0$ for $s = 1$.

On the other hand, one of the key concepts addressed in this study is the q- analogue of a mathematical expression, which involves generalizing the expression by introducing a parameter q. As $q \to 1$, the q-analogue naturally reduces to its classical method. Although the origins of q-calculus can be traced back to the work of Euler, it has become a more vibrant and actively researched field in recent years. q-calculus has attracted attention due to its wide range of applications in mathematics, physics, and engineering. It finds extensive use in various branches of mathematics, including approximation theory, combinatorics, quantum algebra, special functions, operator theory, hypergeometric functions, and beyond. The q-Cesàro matrix $C^q = (c_{nv}^q)$, which is one of the basic concepts of this study, has recently been defined by Aktuğlu and Bekar (2011) as follows:

$$c_{nv}^q = \begin{cases} \dfrac{q^v}{[n+1]_q}, & 0 \le v \le n \\ 0, & v > n \end{cases}$$

where $[n]_q$ is the q-analogue of a non-negative number n and identified by

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & q \epsilon R^+ - \{1\} \\ n, & q = 1. \end{cases}$$

In fact, the q-Cesàro matrix has been used in some previous studies (Çınar & Et, 2020; Erdem, 2024; Yaying et al. 2025; Yaying et al., 2021). While these studies mainly focus on classical or statistical forms of q-Cesàro summability, the present paper differs from the existing literature in that it introduces an absolute summability method generated by the transformation matrix obtained by q-Cesàro matrix, which provides a stronger and more flexible convergence framework. Absolute summability often yields results that cannot be obtained through other q-summability methods, making the method particularly effective in situations where enhanced convergence, stabilization, or smoothing is required. Moreover, the absolute q-Cesàro method has potential applications in areas such as sequence transformations, approximation processes, adaptive smoothing, and signal or data analysis, where stronger convergence criteria play a significant analytical role.

At this point, it should be emphasized that q-Cesàro offers a more adaptable convergence analysis framework than the classical Cesàro approach. This flexibility arises from the role of the parameter q, which functions as a deformation factor in the summability process. As $q \to 1$, the classical Cesàro summability method is recovered. For values of q different from 1, however, the weights of the terms are redistributed, which frequently ensures convergence in situations where the classical method does not succeed. Because of this property, q-Cesàro techniques find relevance in several areas, including statistical convergence, approximation processes, and the theory of q-difference equations.

To highlight the distinction between q-analogue summability and its classical counterparts, consider the sequence

$$x_n = \frac{(-1)^n}{\sqrt{n+1}}.$$

This sequence is not classically convergent, but it becomes summable for $0 < q < 1$ under a suitable q-Cesàro summability method (see Figure 1 for $q = 0.7, s = 1$). Beyond these theoretical aspects, applications to Fourier series and q-difference equations reveal the practical importance of q-Cesàro methods**.** They provide a refined tool for capturing subtle forms of convergence, including weak or statistical types. Furthermore, in engineering contexts such as signal processing and data compression, it is often desirable to suppress noise or smooth irregular fluctuations. While traditional Cesàro means apply uniform averaging, this may not suffice for highly variable data. In contrast, the q-Cesàro framework includes an adjustable q parameter that allows the researcher to balance smoothing and the preservation of local details, offering greater versatility. In future work, the potential use of q-Cesàro summability in adaptive signal filtering, noise reduction, and compression algorithms deserves further attention. Figure 1 already suggests that when $q < 1$, q-Cesàro method smoothing outperforms the classical scheme, especially in the presence of rapid oscillations.

Nevertheless, some limitations should be acknowledged. Calculations based on q-calculus such as q-differences or q-summability operators are typically more complicated than classical formulations, and proofs involving q-Cesàro matrices often require considerable technical effort. Moreover, the parameter q is not universal; its choice crucially influences the outcome. Therefore, q must be selected carefully according to the nature of the application, rather than chosen arbitrarily.

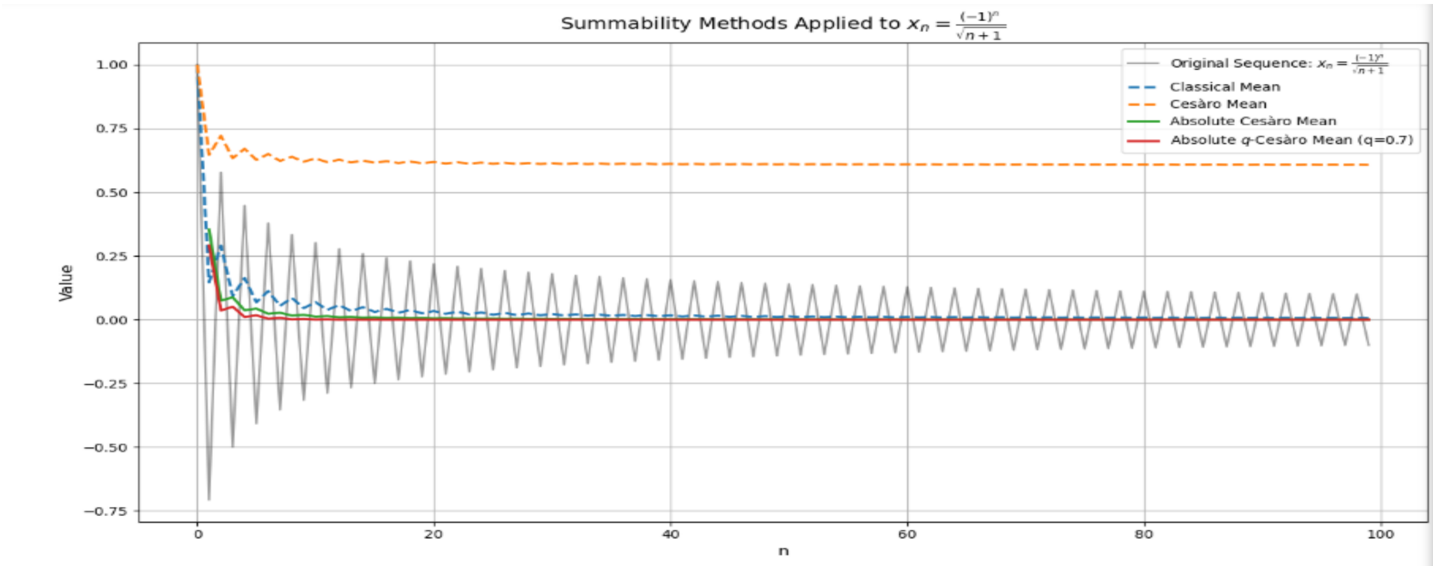

Figure 1: Summability methods applied to $x_n = \frac{(-1)^n}{\sqrt{n+1}}$

Figure 1 shows how the absolute q-Cesàro mean, the classical mean, the q-Cesàro summability, and the absolute Cesàro mean methods behave when applied to an oscillatory sequence $x_n = \frac{(-1)^n}{\sqrt{n+1}}$. The figure demonstrates that classical summability methods are less effective in mitigating oscillatory behavior, whereas the q-Cesàro method accelerates stabilization.

Let $b = (b_j)$ denote the sequence of partial sums of the series $\sum u_i$, and let $\varphi = (\varphi_j)$ be any sequence of positive real numbers with $1 \leq s < \infty$. Following Sarıgöl (2010), the series $\sum u_i$ is said to be summable $|\Lambda, \varphi|_s$, if

$$\sum_{j=1}^{\infty} \varphi_j^{s-1} \left|\Lambda_j(b) - \Lambda_{j-1}(b)\right|^s < \infty.$$

This summability method $|\Lambda, \varphi|_s$ is highly general and encompasses many well-known absolute summability methods as special cases, depending on the choice of the matrix Λ and the sequence φ. For example, if one takes the triangle matrix T instead of Λ, the summability method $|\mathrm{T}, \varphi|_s$ is immediately obtained (Gökçe, 2022). Similarly, choosing the Euler or Fibonacci matrices etc. yields the summability methods $|E^r, \varphi|_s$, $|F, \varphi|_s$ (Gökçe & Sarıgöl, 2020a, 2020b), respectively. Further examples and related discussions can be found in (see also (Gökçe, 2022, 2024; Gökçe & Sarıgöl, 2019)).

Finally, before moving on to the main sections, let us recall some lemmas that will be used in the proofs:

Lemma 1.1. (Stieglitz &Tietz, 1977) *Let* $1 < s < \infty$. $\Lambda \in (l_s, l)$ *if and only if*

$$sup\left\{\sum_{j=0}^{\infty}\left|\sum_{n\in N}\lambda_{nj}\right|^{s^*} : N \subset \mathbb{N}\ finite\right\} < \infty.$$

Lemma 1.2. (Sarıgöl, 2013) *Let* $\Lambda = (\lambda_{ji})$ *be an infinite matrix with complex components,* $\varrho = (\varrho_i)$ *be a bounded sequence of positive numbers. If* $W_\varrho[\Lambda] < \infty$ *or* $L_\varrho[\Lambda] < \infty$, *then*

$$(2m)^{-2} W_\varrho[\Lambda] \leq L_\varrho[\Lambda] \leq W_v[\Lambda],$$

where $m = max\{1, 2^{M-1}\}, M = sup_i\, \varrho_i$.

$$W_\varrho[\Lambda] = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}|\lambda_{ji}|\right)^{\varrho_i}$$

and

$$L_\varrho[\Lambda] = sup\left\{\sum_{i=0}^{\infty}\left|\sum_{j\in G}\lambda_{ji}\right|^{\varrho_i} : G \subset \mathbb{N}\ finite\right\}.$$

Lemma 1.3. (Maddox, 1970) $\Lambda \in (l, l_s)$ *if and only if*

$$\sup_j \sum_{n=0}^{\infty} |\lambda_{nj}|^s < \infty,$$

where $1 \leq s < \infty$.

Lemma 1.4. (Stieglitz & Tietz, 1977) $\Lambda \in (l, l_\infty)$ *if and only if*

$$\sup_{n,j}|\lambda_{nj}| < \infty.$$

2. Main Results

In this part of the paper, we introduce the absolute q-Cesàro summability method, which combines the notion of absolute summability with the transformation matrix generated by the q-Cesàro matrix. To obtain this method, let us take the sequence $\sum u_i$ and its partial sums b_j. Then we get

$$\Lambda_n(b) = \sum_{i=0}^{n} c_{ni}^{q} b_i = \sum_{j=0}^{n} u_j \sum_{i=j}^{n} \frac{q^i}{[n+1]_q} = \sum_{j=0}^{n} u_j \,(1 - \frac{[j]_q}{[n+1]_q})$$

and so,

$$\Delta\Lambda_n(b) = \sum_{j=0}^{n} u_j \left(1 - \frac{[j]_q}{[n+1]_q}\right) - \sum_{j=0}^{n-1} u_j \left(1 - \frac{[j]_q}{[n]_q}\right)$$

$$= \sum_{j=1}^{n} \frac{q^n [j]_q}{[n]_q [n+1]_q} u_j, \quad n > 0, \Delta\Lambda_0(b) = u_0.$$

If

$$\sum_{n=0}^{\infty} \varphi_n^{s-1} \mid \Delta\Lambda_n(b)|^s < \infty,$$

the series $\sum u_i$ is said to be summable by the method $|C^q, \varphi|_s$. Also, considering the transformation sequence (T_n), it can be written that the series $\sum u_i$ is summable by $|C^q,\ \varphi|_s$ if and only if $(T_n) \epsilon\, l_s$. Here

$$T_n = \varphi_n^{1/s^*} \sum_{j=1}^{n} \frac{q^n [j]_q}{[n]_q [n+1]_q} u_j, \quad n > 0,$$

$$T_0 = \varphi_0^{1/s^*} u_0.$$

By making a few calculations, it can be seen that the inverse transformation of the transformation sequence (T_n) is as follows:

$$u_n = T_n \frac{[n+1]_q}{\varphi_n^{1/s^*} q^n} - T_{n-1} \frac{[n-1]_q}{\varphi_{n-1}^{1/s^*} q^{n-1}}, n > 0, \qquad (1)$$

$$u_0 = \varphi_0^{-1/s^*} T_0.$$

It is noted in case of $q = 1$ and $\varphi_n = n$ the summability method $|C^q,\ \varphi_j|_s$ reduces to the well known classical absolute Cesàro method $|C, 1|_s$, (Rhoades, 1998).

For simplicity, throughout the rest of the article, it will be used that

$$[j]_p[j+1]_q - [j+1]_p[j]_q = \Delta_{pq}(j).$$

In the following two theorems, we establish the equivalence criteria that characterize when $|C^q,\varphi|_s \Rightarrow |C^p,\delta|$ and $|C^q,\varphi| \Rightarrow |C^p,\delta|_s$.

Theorem 2.1. *Let* $1 < s < \infty$, $\varphi = (\varphi_n)$ *and* $\delta = (\delta_n)$ *be two sequences for positive numbers. Every series summable by the method* $|C^q,\varphi|_s$ *is also summable by the method* $|C^p,\delta|$, *i.e.,* $|C^q,\varphi|_s \Rightarrow |C^p,\delta|$, *if and only if*

$$\sum_{j=0}^{\infty}\left(\left|\varphi_j^{-1/s^*}\frac{p^j[j+1]_q}{q^j[j+1]_p}\right| + \left|\frac{\sigma_{j+1}^{(p)}}{\varphi_j^{1/s^*}q^j}\Delta_{pq}(j).\right|\right)^{s^*} < \infty \tag{2}$$

where

$$\sigma_{j+1}^{(p)} = \begin{cases} \dfrac{1}{j+1}, & p = 1 \\ \dfrac{p^{j+1}}{[j+1]_p}, & p < 1 \\ \dfrac{1}{[j+1]_p}, & p > 1. \end{cases}$$

Proof. Let T_n and t_n be the transformation sequences of $|C^q,\varphi|_s$ and $|C^p,\delta|$ means of series $\sum u_i$, respectively. It follows from (1) that, for all $n > 0$,

$$t_n = \sum_{j=1}^{n}\frac{p^n[j]_p}{[n]_p[n+1]_p}u_j = \sum_{j=1}^{n}\frac{p^n[j]_p}{[n]_p[n+1]_p}\left(T_j\frac{[j+1]_q}{\varphi_j^{1/s^*}q^j} - T_{j-1}\frac{[j-1]_q}{\varphi_{j-1}^{1/s^*}q^{j-1}}\right)$$

$$= \sum_{j=1}^{n}\frac{p^n[j]_p}{[n]_p[n+1]_p}T_j\frac{[j+1]_q}{\varphi_j^{1/s^*}q^j} - \sum_{j=0}^{n-1}\frac{p^n[j+1]_p}{[n]_p[n+1]_p}T_j\frac{[j]_q}{\varphi_j^{1/s^*}q^j}$$

$$= \varphi_n^{-1/s^*}\frac{p^n[n+1]_q}{q^n[n+1]_p}T_n + \sum_{j=1}^{n-1}\frac{\varphi_j^{-1/s^*}p^n}{q^j[n]_p[n+1]_p}\left([j]_p[j+1]_q - [j+1]_p[j]_q\right)T_j,$$

$$t_0 = \varphi_0^{-1/s^*}T_0.$$

So, it can be written that

$$t_n = \sum_{j=0}^{n}a_{nj}T_j \quad ,$$

where

$$a_{nj}=\begin{cases}\varphi_n^{-1/s^*}\dfrac{p^n[n+1]_q}{q^n[n+1]_p}, & j=n\\ \dfrac{p^n}{\varphi_j^{1/s^*}q^j[n]_p[n+1]_p}\Delta_{pq}(j), & 1\le j\le n-1\\ 0, & j>n.\end{cases}$$

So, it can be immediately seen that $(t_n)\epsilon l$ whenever $(T_n)\epsilon l_s$ if and only if $A\in(l_s,l)$. Also, if the series $\sum_{n=j+1}^{\infty}\frac{p^n}{[n]_p[n+1]_p}$ can be considered a telescopic series as follows

$$\sum_{n=j+1}^{\infty}\frac{p^n}{[n]_p[n+1]_p}=\begin{cases}\displaystyle\sum_{n=j+1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right), & p=1\\ \displaystyle\sum_{n=j+1}^{\infty}\left(\frac{1}{1-p^n}-\frac{1}{1-p^{n+1}}\right), & p<1 \text{ or } p>1,\end{cases}$$

the sum of the series is found as follows for each value of p:

$$\sigma_{j+1}^{(p)}=\begin{cases}\dfrac{1}{j+1}, & p=1\\ \dfrac{p^{j+1}}{[j+1]_p}, & p<1\\ \dfrac{1}{[j+1]_p}, & p>1.\end{cases}$$

Hence, by applying Lemma 1.1 and Lemma 1.2 to the matrix A, condition (2) is obtained which concludes the proof.

Theorem 2.2. *Let $1\le s<\infty$, $\varphi=(\varphi_n)$ and $\delta=(\delta_n)$ be two sequences for positive numbers. Every series summable by the method $|C^q,\varphi|$ is also summable by the method $|C^p,\delta|_s$, i.e., $|C^q,\varphi|\Rightarrow|C^p,\delta|_s$, if and only if*

$$\sup_j\left(\left|\delta_j^{1/s^*}\frac{p^j[j+1]_q}{q^j[j+1]_p}\right|^s+\sum_{n=j+1}^{\infty}\left|\frac{\delta_n^{1/s^*}p^n}{q^j[n]_p[n+1]_p}\Delta_{pq}(j)\right|^s\right)<\infty.$$

Proof. The proof follows similar steps to Theorem 2.1, therefore it is left to the reader.

Theorem 2.3. *Let* $1 \leq s < \infty$, $\varphi = (\varphi_n)$ *and* $\delta = (\delta_n)$ *be two sequences for positive numbers. In order that, every summable series by the method* $|C^q, \varphi|_s$ *is also summable by the method* $|C^p, \delta|_s$*, i.e.,* $|C^q, \varphi|_s \Rightarrow |C^p, \delta|_s$ *, the conditions*

$$\sup_n \left| \frac{\delta_n^{1/s^*}}{\varphi_n^{1/s^*}} \frac{p^n [n+1]_q}{q^n [n+1]_p} \right| < \infty, \tag{3}$$

$$\sup_n \sup_{j \leq n-1} \left| \frac{\delta_n^{1/s^*} p^n}{\varphi_j^{1/s^*} q^j [n]_p [n+1]_p} \Delta_{pq}(j) \right| < \infty \tag{4}$$

are necessary. Moreover, if the conditions

$$\sum_{n=j+1}^{\infty} \left(\sum_{j=0}^{n-1} \left| \frac{\delta_n^{\frac{1}{s^*}} p^n}{\varphi_j^{1/s^*} q^j [n]_p [n+1]_p} \Delta_{pq}(j) \right|^{s^*} \right)^{\frac{s}{s^*}} = O(1) \tag{5}$$

$$\delta_n^{1/s^*} \frac{p^n [n+1]_q}{\varphi_n^{1/s^*} q^n [n+1]_p} = O(1) \tag{6}$$

hold, then it is said that $|C^q, \varphi|_s \Rightarrow |C^p, \delta|_s$.

Proof. Assume that T_n and $\hat{t}_n$ are the transformation sequences of $|C^q, \varphi|_s$ and $|C^p, \delta|_s$ means of series $\sum u_i$, respectively.

Using the inverse transformation of T_n, it is obtained that for all $n \geq 1$:

$$\begin{aligned}
\hat{t}_n &= \delta_n^{1/s^*} \sum_{j=1}^{n} \frac{p^n [j]_p}{[n]_p [n+1]_p} u_j = \delta_n^{1/s^*} \sum_{j=1}^{n} \frac{p^n [j]_p}{[n]_p [n+1]_p} \left(T_j \frac{[j+1]_q}{\varphi_j^{1/s^*} q^j} - T_{j-1} \frac{[j-1]_q}{\varphi_{j-1}^{1/s^*} q^{j-1}} \right) \\
&= \delta_n^{1/s^*} \sum_{j=1}^{n} \frac{p^n [j]_p}{[n]_p [n+1]_p} T_j \frac{[j+1]_q}{\varphi_j^{1/s^*} q^j} - \delta_n^{1/s^*} \sum_{j=0}^{n-1} \frac{p^n [j+1]_p}{[n]_p [n+1]_p} T_j \frac{[j]_q}{\varphi_j^{1/s^*} q^j} \\
&= \frac{\delta_n^{1/s^*}}{\varphi_n^{1/s^*}} \frac{p^n [n+1]_q}{q^n [n+1]_p} T_n + \delta_n^{1/s^*} \sum_{j=1}^{n-1} \frac{p^n}{\varphi_j^{1/s^*} q^j [n]_p [n+1]_p} \left([j]_p [j+1]_q - [j+1]_p [j]_q \right) T_j,
\end{aligned}$$

$$t_0 = \delta_0^{1/s^*} \varphi_0^{-1/s^*} T_0.$$

So, it can be written that

$$\hat{t}_n = \sum_{j=0}^{n} b_{nj}\, T_j,$$

where

$$b_{nj} = \begin{cases} \dfrac{\delta_n^{1/s^*}}{\varphi_n^{1/s^*}} \dfrac{p^n[n+1]_q}{q^n[n+1]_p}, & j = n \\ \dfrac{\delta_n^{1/s^*} p^n}{\varphi_j^{1/s^*} q^j [n]_p [n+1]_p} \Delta_{pq}(j)\,, & 1 \le j \le n-1 \\ 0, & j > n. \end{cases}$$

Here, it can be written that $|C^q,\varphi|_s \Rightarrow |C^p,\delta|_s$ is equal to $B \in (l_s, l_s)$. Since $l \subset l_s \subset l_\infty$, B also belongs to the matrix class (l, l_∞). Hence, with Lemma 1.4, the conditions (3) and (4) are obtained. On the other part, it follows from Minkowski and Hölder inequalities that

$$\left(\sum_{n=0}^{\infty} |\hat{t}_n|^s\right)^{1/s} = \left(\sum_{n=0}^{\infty} \delta_n^{s-1} \left|\sum_{j=1}^{n-1} \frac{p^n}{\varphi_j^{1/s^*} q^j [n]_p [n+1]_p} \Delta_{pq}(j)\, T_j + \frac{p^n[n+1]_q}{\varphi_n^{1/s^*} q^n [n+1]_p} T_n\right|^s\right)^{1/s}$$

$$\le \left(\sum_{n=0}^{\infty} \delta_n^{s-1} \left|\sum_{j=1}^{n-1} \frac{p^n \Delta_{pq}(j)}{\varphi_j^{1/s^*} q^j [n]_p [n+1]_p} T_j\right|^s\right)^{1/s} + \left(\sum_{n=0}^{\infty} \delta_n^{s-1} \left|\frac{p^n[n+1]_q}{\varphi_n^{1/s^*} q^n [n+1]_p} T_n\right|^s\right)^{1/s}$$

$$\le \left(\sum_{j=1}^{\infty} |T_j|^s \sum_{n=j+1}^{\infty} \left(\sum_{j=0}^{n-1} \left|\frac{\delta_n^{\frac{1}{s^*}} p^n\, \Delta_{pq}(j)}{\varphi_j^{1/s^*} q^j [n]_p [n+1]_p}\right|^{s^*}\right)^{\frac{s}{s^*}}\right)^{\frac{1}{s}} + \left(\sum_{n=0}^{\infty} \delta_n^{s-1} \left|\frac{p^n[n+1]_q}{\varphi_n^{1/s^*} q^n [n+1]_p} T_n\right|^s\right)^{\frac{1}{s}}$$

$$= T^{(1)} + T^{(2)}.$$

If $T^{(i)} = O(1), i = 1,2$, then it is written that $(\hat{t}_n) \epsilon l_s$, or equivalently, if the conditions (5) and (6) hold, $|C^q,\varphi|_s \Rightarrow |C^p,\delta|_s$. This concludes the proof.

3. Conclusion

In this study, it have been introduced the absolute summability method $|C^q,\varphi|_s$ generated by a transformation matrix derived from the q-Cesàro matrix. The main purpose has been to establish the necessary and sufficient conditions for the inclusion relations $|C^q,\varphi|_s \Rightarrow |C^p,\delta|$, $|C^q,\varphi| \Rightarrow |C^p,\delta|_s$ and $|C^q,\varphi|_s \Rightarrow$

$|C^p, \delta|_s$ for $1 \leq s < \infty$. The results obtained in this work are expected to enrich the theory of summability by providing a broader framework that connects q-Cesàro methods with other summability approaches. Furthermore, these findings may stimulate future research on extending q-Cesàro methods to more generalized sequence structures, exploring their relationships with modular summability and statistical convergence, and investigating their implications within sequence spaces and operator theory. In addition to their theoretical significance, the methods discussed here may find potential applications in various engineering contexts, such as signal smoothing, adaptive data compression, and other areas where summability techniques play a key analytical role.

Acknowledgement: The author would like to thank the referees for their valuable comments and fruitful suggestions that improved the readability of the manuscript.

Availability of Data and Materials: No data were used to support the findings of the study.

Ethical Declarations: Not applicable.

Conflicts of Interest: The author declares no conflicts of interest.

Funding: There was no funding support for this study.

Authors' Contributions: The author contributed solely to this work and has read and approved the final version of the manuscript.

Generative AI Declaration: The author confirms that generative AI tools were not used for the creation of any substantive content, mathematical proofs, or results in this study.

References:

Aktuğlu, H. & Bekar, S. (2011). q-Cesàro Matrix and q-Statistical Convergence. *Journal of Computational and Applied Mathematics*, 235 (16), 4717-4723.

Bilgin Ellidokuzoğlu, H., Demiriz, S., & Köseoğlu, A. (2018). On the paranormed binomial sequence spaces. *Universal Journal of Mathematics and Applications*, 1 (3), 137-147.

Çınar, M. & Et, M., (2020). q-double Cesàro matrices and q-statistical convergence of double sequences. *National Academy Science Letters*, 43, 73-76.

Erdem, S., (2024). On the q-Cesàro bounded double sequence space, *Mathematical Sciences and Applications E-Notes*, 12 (3), 145-154.

Gökçe, F. (2024). Paranormed spaces of absolute tribonacci summable series and some matrix transformations. *Dera Natung Government College Research Journal*, 9 (1), 13-27.

Gökçe, F. (2022). Compact matrix operators on Banach space of absolutely k-summable series. *Turkish Journal of Mathematics*, 46 (3), 1004-1019.

Gökçe, F. & Sarıgöl, M.A. (2020a). Some matrix and compact operators of the absolute Fibonacci series spaces. *Kragujevac Journal of Mathematics*, 44 (2), 273–286.

Gökçe, F. & Sarıgöl, M.A. (2020b). On absolute Euler spaces and related matrix operators. *Proceedings of the National Academy of Sciences, India. Section A. Physical Sciences*, 90(5), 769-775.

Gökçe, F. & Sarıgöl, M.A. (2019). Generalization of the absolute Cesàro space and some matrix transformations. *Numerical Functional Analysis and Optimization*, 40, 1039-1052.

Gürdal, M. & Yamancı, U. (2015). Statistical convergence and some questions of operator theory. *Dynamic Systems and Applications*, 24(3), 305-311.

Huban, M.B. & Gürdal, M. (2021). Wijsman lacunary invariant statistical convergence for triple sequences via Orlicz function. *Journal of Classical Analysis*, 17(2), 119-128.

Indumathi A., Esi A. & Subramanian N. (2023). On Gradual Borel Summability Method of Rough Convergence of Triple Sequences of Beta Stancu Operators, *Dera Natung Government College Research Journal*, 8 (1), 14-29.

Kara, M. I., & Bayrakdar, M. A. (2021). A study on matrix domain of Riesz-Euler totient matrix in the space of p-absolutely summable sequences. *Communications in Advanced Mathematical Sciences*, 4 (1), 14-25.

Kişi, Ö., Gürdal, M. & Çetin, S. (2025). On Some Properties of Fibonacci Almost I-Statistical Convergence of Fuzzy Variables in Credibility Space. *Boletim da Sociedade Paranaense de Matemática*, 43 (2), 16-30.

Kişi, O. & Gürdal, M. (2022). On deferred Cesàro summability and statistical convergence for the sets of triple sequences. *Annals of Fuzzy Mathematics and Informatics*, 24, 115-127.

Kişi, Ö. & Erhan, G. (2018). I-Cesàro summability of a sequence of order α of random variables in probability. *Fundamental Journal of Mathematics and Applications*, 1 (2), 157-161.

Maddox, I.J. (1970). Elements of functional analysis. *Cambridge University Press*, London, New York.

B.E. Rhoades, (1998). Absolute comparison theorems for double weighted mean and double Cesàro means, *Mathematica Slovaca*, 48, 285-291.

Sarıgöl, M.A. (2013). An inequality for matrix operators and its applications. *Journal of Classical Analysis*, 2, 145-150.

Sarıgöl, M.A. (2010). On the local properties of factored Fourier series. *Applied Mathematics Computations*, 216 (11), 3386-3390.

Savaş, E., Kişi, Ö. & Gürdal, M. (2022). On statistical convergence in credibility space. *Numerical Functional Analysis and Optimization*, 43 (8), 987-1008.

Stieglitz, M. & Tietz, H. (1977). Matrix transformationen von Folgenraumen. Eine Ergebnisubersicht. *Mathematische Zeitschrift*, 154 (1), 1-16.

Yamancı, U. & Gürdal, M. (2016). Statistical convergence and operators on Fock space. *New York Journal of Mathematics*, 22, 199-207.

Yaying, T., Mohiuddine, S. A., Aljedani, J. (2025). Exploring the q-analogue of Fibonacci sequence spaces associated with c and c_0. *AIMS Mathematics*, 10 (1), 634-653.

Yaying, T., Hazarika, B., Mursaleen, M. (2021). On sequence space derived by the domain of q-Cesàro matrix in ℓ_p space and the associated operator ideal. *Journal of Mathematical Analysis and Applications*, 493 (1), 124453.