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Research Article

On the domain of the Pell-Lucas matrix in the spaces c and c_0

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Abstract: In this study, we introduce new Banach sequence spaces $c(\Theta)$ and $c_0(\Theta)$, defined via a regular infinite matrix $\Theta = (\lambda_{nk})$, where

$$\Theta_{nk} = \begin{cases} \dfrac{2\lambda_k}{3\lambda_n + \lambda_{n-1}} & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

and λ_k represents the k^{th} element of Pell-Lucas sequence. The study primarily focuses on exploring the fundamental properties and inclusion relationships of the corresponding sequence spaces, establishing a Schauder basis, and determining their α-, β-, and γ-duals. In addition, we characterize the connections between the newly defned matrix classes and classical sequence spaces. We also examine the compactness of matrix operators within these associated sequence spaces.

Keywords: Pell-Lucas numbers, Sequence space, Schauder basis, Compactness.

MSC 2020: 46A45, 46B45, 11B83, 46B50.

1. Introduction

Matrix operators play a fundamental role in functional analysis and operator theory, particularly in the study of sequence spaces. They provide a framework for examining how sequences are transformed under various linear mappings. Among the structural properties of such operators, compactness is of central importance, especially in spectral theory, where it enables the reduction of intricate infinite-dimensional problems to finite-dimensional analogues. Consequently, compact matrix operators are closely related to summability theory, approximation processes, and the analysis of operator equations. Let $\mathbb{N} = 1,2,3, \ldots$ denote the set of natural numbers, and let ω represent the space of all real-valued sequences. Within this framework, we define several important subspaces:

- ℓ_p: The set of all sequences that are absolutely p-summable,
- ℓ_∞: the space of all bounded sequences,

- c_0: the space of sequences that converge to zero,
- c: the space of all convergent sequences.

The spaces ℓ_∞, c and c_0 are Banach spaces equipped with the supremum norm:

$$\|x\|_\infty = \sup_k |x_k|.$$

Additionally, for $1 \leq p < \infty$, the space ℓ_p forms a Banach space, with the norm:

$$\|x\|_{\ell_p} = \left(\sum_k |x_k|^p\right)^{1/p}.$$

A Banach space $\mathfrak{X}$ is called a BK-space if the map that takes a sequence to its n-th term, $x \mapsto x_n$, is continuous for every n. Examples include ℓ_p and ℓ_∞. Given two sequence spaces $\mathfrak{X}$ and $\mathfrak{Y}$, and an infinite real matrix $A = (a_{nk})$, we denote the n-th row as A_n. The matrix A maps $\mathfrak{X}$ to $\mathfrak{Y}$ if for every sequence $x = (x_k)$, the sequence

$$Ax = \{A_n x\}_{n=0}^{\infty} \text{ with } A_n x = \sum_k a_{nk} x_k \tag{1}$$

belongs to $\mathfrak{Y}$. The domain of A is the set $\mathfrak{X}_A = x \in \omega \colon Ax \in \mathfrak{Y}$. The notation $(\mathfrak{X}, \mathfrak{Y})$ designates the family of all matrices A mapping from $\mathfrak{X}$ to $\mathfrak{Y}$. Thus, $A \in (\mathfrak{X}, \mathfrak{Y})$ precisely when the series in equation (1) converges for every $n \in \mathbb{N}$ and each $x \in \mathfrak{X}$, which guarantees that $Ax \in \mathfrak{Y}$ for all $x \in \mathfrak{X}$.

Earlier, (Erdem et al., 2024) developed the Motzkin matrix spaces $c(\mathcal{M})$ and $c_0(\mathcal{M})$, revealing their internal structure and introducing the concept of the Motzkin core. Subsequently, (Demiriz et al., 2025) introduced the BK-spaces $\ell_p(G)$ and $\ell_\infty(G)$ using generalized Motzkin matrices, establishing their basis properties. Based on these findings, (Erdem, 2024a) explored $\ell_p(\mathcal{M})$ spaces along with compact operators, while (Erdem, 2024b) extended the theory to Schröder–Catalan matrix spaces with similar results.

2. Pell-Lucas Matrix Operator and Pell-Lucas Sequence Spaces

Pell sequence is historically linked to the English mathematician John Pell (1611–1685), while its companion sequence, known as the Pell–Lucas sequence, is associated with the work of the French mathematician Édouard Lucas (1842–1891). Comprehensive studies of these sequences are available in (Horadam, 1994; Bicknell, 1975; Dasdemir, 2011). It was demonstrated in (Dasdemir, 2011; Ercolano, 1979) that Pell numbers can be expressed in matrix form. In addition, Atabey et al. (Atabey et al., 2025) introduced a sequence space generated by Pell numbers. Both the Pell and Pell–Lucas sequences possess well-known recursive characterizations, among several other equivalent definitions. In particular, the Pell–Lucas numbers satisfy the recurrence relation

$$\lambda_k = 2\lambda_{k-1} + \lambda_{k-2}$$

with initial values $\lambda_0 = 2, \lambda_1 = 2$. According to this formula, the first few Pell-Lucas numbers are $2, 2, 6, 14, 34, 82, 198, 478, \ldots$. We next examine for the negative Pell-Lucas sequence, defined by $\lambda_{-k} = (-1)^k \lambda_k\}$.

We can easily derive the relation

$$\sum_{s=0}^{k} \lambda_s = \frac{3\lambda_k + \lambda_{k-1}}{2} \tag{2}$$

For a nonnegative integer k, let λ_k represent the k-th Pell-Lucas number. Consider the matrix $\Theta = (\Theta_{nk})$, defined by

$$\Theta_{nk} = \begin{cases} \dfrac{2\lambda_k}{3\lambda_n + \lambda_{n-1}} & 0 \le k \le n, \\ 0, & k > n, \end{cases}$$

where $n, k = 1,2, \ldots$.

$$\Theta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \frac{4}{8} & \frac{4}{8} & 0 & 0 & 0 & 0 & \ldots \\ \frac{4}{20} & \frac{4}{20} & \frac{12}{20} & 0 & 0 & 0 & \ldots \\ \frac{4}{48} & \frac{4}{48} & \frac{12}{48} & \frac{28}{48} & 0 & 0 & \ldots \\ \frac{4}{116} & \frac{4}{116} & \frac{12}{116} & \frac{28}{116} & \frac{68}{116} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is obvious that Θ is triangular. Therefore, the Θ-transform of a sequence $b' = (b'_k)$ is expressed as

$$\Omega'_n = (\Theta b')_n = \frac{2\sum_{k=0}^{n} \lambda_k b'_k}{3\lambda_n + \lambda_{n-1}}. \tag{3}$$

Lemma 2.1. (Peterson, 1998) *An infinite matrix $\mathcal{B} = (b_{n,k})$ qualifies as regular if and only if each of the conditions listed below is satisfied.*

(a) $\sup_{n\in\mathbb{N}} \sum_k |b_{nk}| < \infty$;

(b) $\lim_{n\to\infty} \sum_k b_{nk} = 1$;

(c) $\lim_{n\to\infty} b_{nk} = 0$.

Theorem 2.2. *Pell-Lucas matrix* $\Theta = (\Theta_{nk})$ *is regular.*

Proof. We have

$$\sum_k |\Theta_{n,k}| = \sum_k \Theta_{n,k} = \sum_{k=0}^{n} \frac{2\lambda_k}{3\lambda_n + \lambda_{n-1}} = 1$$

$\therefore \lim_{n\to\infty} \sum_k \Theta_{n,k} = 1$, hence the condition (b). Additionally, condition (a) is satisfied easily.

Again, $\lim_{n\to\infty} \Theta_{n,k} = \lim_{n\to\infty} \frac{2\lambda_k}{3\lambda_n+\lambda_{n-1}} = 2\lambda_k \lim_{n\to\infty} \frac{1}{3\lambda_n+\lambda_{n-1}} = 0$, which satisfies condition (c).

3. Pell-Lucas sequence spaces

We now introduce the Pell-Lucas sequence spaces $c(\Theta)$ and $c_0(\Theta)$. A sequence belongs to one of these spaces if and only if its Θ-transform lies in the respective classical sequence space c or c_0.

$$c(\Theta) = \left\{h = (h_k) \in \omega : \lim_{n\to\infty} \frac{2\lambda_k}{3\lambda_n + \lambda_{n-1}} h_k \text{ exists}\right\} (1 \le p < \infty);$$
$$c_0(\Theta) = h = (h_k) \in \omega : \lim_{n\to\infty} \frac{2\lambda_k}{3\lambda_n + \lambda_{n-1}} h_k = 0.$$

We can express $\mathcal{G}(\Theta)$ as $\mathcal{G}_\Theta$, where $\mathcal{G}$ denotes any of the spaces c, c_0.

Theorem 3.1. *The spaces* $c_0(\Theta)$ *and* $c(\Theta)$ *are BK-spaces with the norm defined by*

$$\|b'\|_{c_0(\Theta)} = \|b'\|_{c(\Theta)} = \sup_{n\in\mathbb{N}} |(\Theta b')_n|$$

Proof. The proof is straightforward from Theorem 4.3.12 of Wilansky (Wilansky, 1984).

Theorem 3.2. $c_0(\Theta) \cong c_0$ *and* $c(\Theta) \cong c$.

Proof. We define the mapping as follows:

$$\mathcal{T} : c_0(\Theta) \to c_0 \text{ such that } \mathcal{T}(b') = \Theta b'.$$

From the result $\mathcal{T}(b') = \theta \Longrightarrow b' = 0$, where θ is the zero element of $c_0(\Theta)$. This proves the injectivity of the mapping $\mathcal{T}$. Furthermore, let $\Omega' \in \ell_\infty$ and define the sequence $b' = (b'_k)$ by

$$b'_k = \sum_{l=k-1}^{k} (-1)^{k-l} \frac{3\lambda_l + \lambda_{l-1}}{2\lambda_k} \Omega'_l, (k \in \mathbb{N}). \tag{4}$$

Then

$$\begin{aligned}\lim_{k\to\infty}(\Theta b')_k &= \lim_{k\to\infty}\left(\sum_{l=0}^{k}\frac{2\lambda_l}{3\lambda_k+\lambda_{k-1}}b'_l\right)\\ &= \lim_{k\to\infty}\left(\sum_{l=0}^{k}\frac{2\lambda_l}{3\lambda_k+\lambda_{k-1}}\sum_{j=l-1}^{l}(-1)^{l-j}\frac{3\lambda_l+\lambda_{l-1}}{2k_l}\Omega'_l\right)\\ &= \lim_{k\to\infty}\Omega'_k\\ &= 0.\end{aligned}$$

Therefore, $b' \in c_0(\Theta)$. Thus, $\mathcal{T}$ is surjective and preserves the norm. Consequently, $c_0(\Theta) \cong c_0$. Other one can be done in similar way.

Theorem 3.3. *The inclusions $c_0 \subset c_0(\Theta)$ and $c \subset c(\Theta)$ are strict.*

Proof. Given that the matrix Θ is regular, the inclusions arise naturally. To establish their strictness, consider the sequence $b' = (1,0,1,0,\dots)$. We can compute the following for this sequence

$$(\Theta b')_n = \sum_{\lambda=0}^{\mu}\frac{2\lambda_k}{3\lambda_n+\lambda_{n-1}}b'_\lambda = \frac{2}{3\lambda_n+\lambda_{n-1}}\left(\lambda_0+\lambda_1+\cdots+\lambda_\mu\right), \text{ where } \mu\in\mathbb{N}.$$

This expression converges, implying that $b' \in c(\Theta)\setminus c$. An analogous approach can be utilized to prove the other case.

Definition 3.4. *A sequence $z=(z_n)$ in a normed space $(Z, \|\cdot\|_Z)$ is defined as a Schauder basis if, for every vector $w \in Z$, there exists a unique sequence of scalars (c_n) such that the following holds*

$$\lim_{n\to\infty}\left\| w - \sum_{n=0}^{n} c_n z_n\right\|_Z = 0.$$

The mapping $k: \mathcal{G}(\Theta) \to \mathcal{G}$, defined in the proof of the previous theorem, establishes an isomorphism between these two spaces. Consequently, the preimage of the basis $\{e^{(k)}\}_{k\in\mathbb{N}}$ in $\mathcal{G}$ serves as a corresponding basis for the newly constructed sequence space $\mathcal{G}(\Theta)$.

Hence, we arrive at the following conclusion:

Theorem 3.5. *Consider the sequence $b^{(k)} = \left(b_n^{(k)}\right)$ define for each fixed $k \in \mathbb{N}$ as follows:*

$$b_n^{(k)} = \begin{cases}\dfrac{(-1)^{n-k}(3\lambda_k+\lambda_{k-1})}{2\lambda_n} & \text{if } n-1\le k\le n,\\ 0 & \text{if } 0\le k<n-1 \text{ or } k>n.\end{cases}$$

Then we have the following results

Corollary 3.6. *The sequence spaces $c_0(\Theta)$ and $c(\Theta)$ are separable.*

Proof. The conclusion follows immediately from Theorem 3.1 and Theorem 3.5.

4. $\alpha-, \beta-, \gamma-$ duals

We identify the $\alpha-$, $\beta-$, and $\gamma-$ duals of the spaces $c_0(\Theta)$ and $c(\Theta)$. Some key results from Stielglitz and Tietz (1977) will be summarized without proofs, as they are crucial for our discussion.

Lemma 4.1. *For $1 < p \leq \infty$, $\mathcal{A} = (a_{nk}) \in (c_0 : \ell_1) = (c : \ell_1)$ iff*

$$\sup_{E \in E_0} \sum_{k=1}^{\infty} \left| \sum_{n \in E} a_{nk} \right| < \infty.$$

Lemma 4.2. *For $1 < p < \infty$, $\mathcal{A} = (a_{nk}) \in (c_0 : c)$ iff*

$$\lim_{n \to \infty} a_{nk} \text{ exists}, \ \forall k \in \mathbb{N}; \tag{5}$$

$$\sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| < \infty. \tag{6}$$

Lemma 4.3. *$\mathcal{A} = (a_{nk}) \in (c : c)$ iff*

$$\sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| < \infty$$

$$\lim_{n \to \infty} a_{nk} \ \textit{exists}, \ \forall k \in \mathbb{N};$$

$$\textit{and} \lim_{n \to \infty} \sum_{k} a_{nk} \ \textit{exists}.$$

Lemma 4.4. *$\mathcal{A} = (a_{nk}) \in (c_0 : \ell_\infty) = (c : \ell_\infty)$ iff*

$$\sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| < \infty.$$

Theorem 4.5. *Consider the infinite matrix $\mathfrak{B}' = (m'_{nk})$* defined by

$$m'_{nk} = \begin{cases} (-1)^{n-k}\dfrac{(3\lambda_k + \lambda_{k-1})}{2\lambda_n} s_n, & \text{if } n-1 \le k \le n \\ 0, & \text{if } k > n \end{cases}$$

and the set $\mathfrak{Z}_1$ *as*

$$\mathfrak{Z}_1 = s = (s_k) \in \omega: \sup_{E \in E_0} \sum_{k=1}^{\infty} \left| \sum_{n \in E} m'_{nk} \right| < \infty$$

then

$$[c_0(\Theta)]^{\alpha} = [c(\Theta)]^{\alpha} = \mathfrak{Z}_1.$$

Proof. By (4), we get

$$\begin{aligned} s_n b'_n &= s_n \left(\sum_{k=n-1}^{n} (-1)^{n-k} \frac{(3\lambda_k + \lambda_{k-1})}{2\lambda_n} \Omega'_k \right) \\ &= \sum_{k=n-1}^{n} \left((-1)^{n-k} \frac{(3\lambda_k + \lambda_{k-1})}{2\lambda_n} \Omega'_k s_n \right) \\ &= \mathfrak{B}'_n \Omega', \ \forall n \in \mathbb{N}. \end{aligned} \tag{7}$$

Therefore, by (7), we get $s \in [c_0(\Theta)]^{\alpha}$ *or* $[c(\Theta)]^{\alpha} \Leftrightarrow \mathfrak{B}' \in (c_0, \ell_1) = (c, \ell_1)$. Thus from Lemma 4.1, we have $[c_0(\Theta)]^{\alpha} = [c(\Theta)]^{\alpha} = \mathfrak{Z}_1$. Hence, the result is proved.

Theorem 4.6. *Consider the sets* $\mathfrak{Z}_2, \mathfrak{Z}_3, \mathfrak{Z}_4$ *as :*

$$\mathfrak{Z}_2 = \left\{ s = (s_k) \in \omega: \sum_k \left| \Delta\left(\frac{s_k}{\lambda_k}\right)\left(\frac{3\lambda_k + \lambda_{k-1}}{2}\right) \right| < \infty \right\};$$

$$\mathfrak{Z}_3 = \left\{ s = (s_k) \in \omega: \left| \sup_k \frac{3\lambda_k + \lambda_{k-1}}{2\lambda_k} s_k \right| < \infty \right\};$$

and

$$\mathfrak{Z}_4 = \left\{ s = (s_k) \in \omega: \lim_{k\to\infty} \frac{3\lambda_k + \lambda_{k-1}}{2\lambda_k} s_k \text{ exists} \right\},$$

where

$$\Delta\left(\frac{s_k}{\lambda_k}\right)\left(\frac{3\lambda_k + \lambda_{k-1}}{2}\right) = \left(\frac{s_\lambda}{\lambda_k} - \frac{s_{k+1}}{\lambda_{k+1}}\right)\left(\frac{3\lambda_k + \lambda_{k-1}}{2}\right).$$

Then,

$$[c_0(\Theta)]^\beta = \mathfrak{Z}_2 \cap \mathfrak{Z}_3, \textit{ and } [c(\Theta)]^\beta = \mathfrak{Z}_2 \cap \mathfrak{Z}_4.$$

Proof. Let $(s_k) \in \omega$ and define $b' = (b'_k)$ according to (4). Now, consider the equality

$$\sum_{k=0}^{n} s_k b'_k = \sum_{k=0}^{n} s_k \left(\sum_{l=k-1}^{k} (-1)^{k-l} \frac{3\lambda_l + \lambda_{l-1}}{2\lambda_k} \Omega'_l \right)$$

$$= \sum_{k=0}^{n-1} \left(\frac{s_k}{\lambda_k} - \frac{s_{k+1}}{\lambda_{k+1}} \right) \left(\frac{3\lambda_k + \lambda_{k-1}}{2} \right) \Omega'_k + \frac{3\lambda_n + \lambda_{n-1}}{2\lambda_n} s_n \Omega'_n$$

$$= \sum_{k=0}^{n-1} \Delta \left(\frac{s_k}{\lambda_k} \right) \left(\frac{3\lambda_k + \lambda_{k-1}}{2} \right) \Omega'_k + \frac{3\lambda_n + \lambda_{n-1}}{2\lambda_n} s_n \Omega'_n \tag{8}$$

$$= (\mathfrak{S}'\Omega')_n \text{ for each } n \in \mathbb{N}, \tag{9}$$

where the matrix $\mathfrak{S}' = (\mathfrak{S}'_{nk})$ is

$$\mathfrak{S}'_{nk} = \begin{cases} \left(\frac{s_k}{\lambda_k} - \frac{s_{k+1}}{\lambda_{k+1}} \right) \left(\frac{3\lambda_k + \lambda_{k-1}}{2\lambda_k} \right) & \text{if } k < n, \\ \left(\frac{3\lambda_n + \lambda_{n-1}}{2\lambda_n} \right) & \text{if } k = n, \\ 0 & \text{if } k > n. \end{cases}$$

It is evident that the columns of the matrix $\mathfrak{S}'$ are convergent, as we have

$$\lim_{n\to\infty} \mathfrak{S}'_{nk} = \left(\frac{s_k}{\lambda_k} - \frac{s_{k+1}}{\lambda_{k+1}} \right) \left(\frac{3\lambda_k + \lambda_{k-1}}{2} \right). \tag{10}$$

By (9) we have $s \in [c_0(\Theta)]^\beta \iff \mathfrak{S}' \in (c_0, c)$.

Hence by (8), (10) and Lemma 4.2, we have

$$\sum_k \left| \lim_{n\to\infty} \Delta \left(\frac{s_k}{\lambda_\lambda} \right) \left(\frac{3\lambda_k + \lambda_{k-1}}{2} \right) \right|^{t'} < \infty$$

and

$$\sup_k \left| \frac{3\lambda_k + \lambda_{k-1}}{2} \right| < \infty.$$

Therefore,

$$[c_0(\Theta)]^{\beta} = \mathfrak{Z}_2 \cap \mathfrak{Z}_3.$$

Similarly, the β-dual of the space $c(\Theta)$ can be determined using Lemma 4.3 and equation (10).

Theorem 4.7. $[c_0(\Theta)]^{\gamma} = [c(\Theta)]^{\gamma} = \mathfrak{Z}_2 \cap \mathfrak{Z}_3$.

Proof. The result can be derived similarly as the previous theorem, this time using Lemma 4.4.

5. Characterization of Matrix Classes

Now, we explore the characteristics of the matrix classes $(c_0(\Theta), \mathcal{G})$ and $(c(\Theta), \mathcal{G})$, where $\mathcal{G}$ can be any of the spaces ℓ_∞, c, or c_0. For convenience, we denote

$$\tilde{c}_{n,k} = \left(\frac{a_{n,k}}{\lambda_k} - \frac{a_{n,k+1}}{\lambda_{k+1}}\right) \cdot \frac{3\lambda_k + \lambda_{k-1}}{2} \tag{11}$$

for all $n, k \in \mathbb{N}$. Additionally, if we have $b', \Omega' \in \omega$ such that $\Omega' = \Theta b'$, we can express the following relationship based on equation (8)

$$\sum_{k=0}^{\xi} a_{n,k} b'_k = \sum_{k=0}^{\xi-1} \tilde{c}_{n,k} \Omega'_k + \frac{3\lambda_\xi + \lambda_{\xi-1}}{2\lambda_\xi} a_{n,\xi} \Omega'_\xi \ (n, \xi \in \mathbb{N}). \tag{12}$$

Next, we outline several conditions to consider as we move forward

$$\sup_n \sum_k |\tilde{c}_{n,k}| < \infty, \tag{13}$$

$$\left(\frac{3\lambda_k + \lambda_{k-1}}{2\lambda_k} a_{n,k}\right)_{k=0}^{\infty} \in \ell_\infty \text{ for every } n \in \mathbb{N}, \tag{14}$$

$$\left(\frac{3\lambda_k + \lambda_{k-1}}{2\lambda_k} a_{n,k}\right)_{k=0}^{\infty} \in c \text{ for every } n \in \mathbb{N}, \tag{15}$$

$$\sup_n \left|\sum_k a_{n,k}\right| < \infty, \tag{16}$$

$$\lim_{n\to\infty} \left|\sum_k a_{n,k}\right| = a \text{ for all } n, k \in \mathbb{N}, \tag{17}$$

$$\lim_{n\to\infty} \left(\sum_k a_{n,k}\right) = 0 \text{ for all } n, k \in \mathbb{N}, \tag{18}$$

$$\lim_{n\to\infty} \tilde{c}_{n,k} = \tilde{a}_k \text{ for } k \in \mathbb{N}, \tag{19}$$

$$\lim_{n\to\infty} \tilde{c}_{n,k} = 0 \text{ for } k \in \mathbb{N}. \tag{20}$$

Using results from (Peterson, 1966) along with Theorem 4.6 and equation (12), we can derive the following conclusions:

Theorem 5.1. *We have*

(1). *The matrix* $\mathcal{A} = (a_{n,k}) \in (c_0(\Theta), \ell_\infty) \Leftrightarrow$ *conditions* (13) *and* (14) *are satisfied.*

(2). *The matrix* $\mathcal{A} = (a_{n,k}) \in (c_0(\Theta), c_0) \Leftrightarrow$ *conditions* (13), (14), *and* (20) *hold.*

(3). *The matrix* $\mathcal{A} = (a_{n,k}) \in (c_0(\Theta), c) \Leftrightarrow$ *conditions* (13), (14), *and* (19) *hold.*

Theorem 5.2.

(1). *The matrix* $\mathcal{A} = (a_{n,k}) \in (c(\Theta), \ell_\infty) \Leftrightarrow$ *conditions* (13), (15), *and* (16) *are satisfied.*

(2). *The matrix* $\mathcal{A} = (a_{n,k}) \in (c(\Theta), c_0) \Leftrightarrow$ *conditions* (13), (15), (18), *and* (20) *hold.*

(3). *The matrix* $\mathcal{A} = (a_{n,k}) \in (c(\Theta), c) \Leftrightarrow$ *conditions* (13), (15), (17), *and* (19) *are hold.*

6. Hausdorff measure of noncompactness

In this section, we derive the necessary and sufficient conditions for an operator to be compact from $c_0(\Theta)$ to a space $\mathcal{H} \in \{c_0, c, \ell_\infty, \ell_1\}$, using the Hausdorff measure of noncompactness. To begin, we revisit key results and notations that are essential to our analysis. For further details on noncompactness, refer to (Dağlı, 2022; Malkowsky & Rakočević, 2000; Rakočević, 1998; Demiriz & Erdem, 2024).

Lemma 6.1. *Let* $\ell_\infty^\beta = c^\beta = c_0^\beta = \ell_1$. *Moreover, for* $\mathcal{G} \in \{\ell_\infty, c, c_0\}$, *the following holds*

$$\|\mathcal{G}\|_{\mathcal{G}}^* = \|\mathcal{G}\|_{\ell_1}.$$

Lemma 6.2. (see Theorem 4.2.8 of (Wilansky, 2000)) *Let* $\mathcal{G}$ *and* $\mathcal{H}$ *be two BK-spaces. Then, the space* $(\mathcal{G}, \mathcal{H})$ *is included in* $B(\mathcal{G}, \mathcal{H})$. *This means that for every operator* A *in* $(\mathcal{G}, \mathcal{H})$, *there exists a corresponding linear operator* $\mathcal{L}_A$ *in* $B(\mathcal{G}, \mathcal{H})$ *defined by* $\mathcal{L}_A g = Ag$ *for all* $g \in \mathcal{G}$.

Lemma 6.3. (see Theorem 2.15 of (Malkowsky & Rakočević, 2000)) *Consider a bounded set* $Q \subset c_0$, *and introduce the operator* $P_s: c_0 \to c_0$ *defined as follows:*

$$P_s(g_0, g_1, g_2, \dots) = (g_0, g_1, g_2, \dots, g_s, 0, 0, \dots) \text{ for any } g = (g_k) \in c_0.$$

Then, we have

$$\chi(Q) = \lim_{s \to \infty} \left(\sup_{g \in Q} \|(I - P_s)(g)\| \right),$$

where I *represents the identity operator on* c_0.

Lemma 6.4. (see Theorem 1.23 of (Malkowsky & Rakočević, 2000)) *Let* $\mathcal{G}$ *be a BK space and* $\phi \subset \mathcal{G}$. *If* $A \in (\mathcal{G}, \mathcal{H})$, *then the norm satisfies*

$$\|\mathcal{L}_A\| = \|A\|_{(\mathcal{G}, \mathcal{H})} = \sup_n \|A_n\|_{\mathcal{G}}^* < \infty.$$

Lemma 6.5. (see Theorem 3.7 of (Mursaleen & Noman, 2010)) *Let* $\mathcal{G}$ *be a BK-space that contains a non-empty set. The following statements are true:*

(a) *If $A \in (\mathcal{G}, c_0)$, then*

$$\|\mathcal{L}_A\|_\chi = \limsup_{n\to\infty} \|A_n\|_{\mathcal{G}}^*,$$

and $\mathcal{L}_A$ is compact $\Leftrightarrow$

$$\lim_{n\to\infty} \|A_n\|_{\mathcal{G}}^* = 0.$$

(b) *If $\mathcal{G}$ has AK and $A \in (\mathcal{G}, c)$, then*

$$\frac{1}{2}\limsup_{n\to\infty} \|A_n - a\|_{\mathcal{G}}^* \leq \|\mathcal{L}_A\|_\chi \leq \limsup_{n\to\infty} \|A_n - a\|_{\mathcal{G}}^*,$$

and $\mathcal{L}_A$ is compact $\Leftrightarrow$

$$\lim_{n\to\infty} \|A_n - a\|_{\mathcal{G}}^* = 0,$$

where $a = (a_k)$ with $a_k = \lim_{n\to\infty} a_{nk}$ for all $k \in \mathbb{N}$.

(c) *If $A \in (\mathcal{G}, \ell_\infty)$, then*

$$0 \leq \|\mathcal{L}_A\|_\chi \leq \limsup_{n\to\infty} \|A_n\|_{\mathcal{G}}^*,$$

and $\mathcal{L}_A$ is compact if

$$\lim_{n\to\infty} \|A_n\|_{\mathcal{G}}^* = 0.$$

For the rest of this paper, E_0 denotes the subset of E consisting of elements of $\mathbb{N}$ that are greater than k.

Lemma 6.6. (see Theorem 3.11 of (Mursaleen & Noman, 2010)) *Let $\mathcal{G}$ be a BK-space containing a non-empty set. If $A \in (\mathcal{G}, \ell_1)$, then*

$$\lim_{k\to\infty} \sup_{E\in E_0} \left(\sum_{n\in E} \|A_n\|_{\mathcal{G}}^* \right) \leq \|\mathcal{L}_A\|_\chi \leq 4 \cdot \lim_{k\to\infty} \sup_{E\in E_0} \left(\sum_{n\in E} \|A_n\|_{\mathcal{G}}^* \right),$$

and

$$\mathcal{L}_A \text{ is compact } \Leftrightarrow \lim_{k\to\infty} \left(\sup_{E\in E_0} \sum_{n\in E} \|A_n\|_{\mathcal{G}}^* \right) = 0.$$

Lemma 6.7. *Consider a sequence space $\mathcal{G}$, with matrices $A = (a_{nk})$ and $\mathfrak{A} = (\tilde{c}_{nk})$ as defined in* (11). *If A is included in the space $(c_0(\Theta), \mathcal{G})$, then it follows that $\mathfrak{A}$ is also part of $(c_0, \mathcal{G})$. Additionally, for any g belonging to $c_0(\Theta)$, there exists a sequence $\tilde{y}$ that corresponds to g such that $Ag = \mathfrak{A}\tilde{y}$.*

Theorem 6.8. *The following statements hold:*

(a) $A \in (c_0(\Theta), c_0)$, *then*

$$\|\mathcal{L}_A\|_\chi = \limsup_{n\to\infty} \sum_k |\tilde{c}_{nk}|.$$

(b) *If* $A \in (c_0(\Theta), c)$, *then*

$$\frac{1}{2} \limsup_{n\to\infty} \sum_k |\tilde{c}_{nk} - \alpha_k| \le \|\mathcal{L}_A\|_\chi \le \limsup_{n\to\infty} \sum_k |\tilde{c}_{nk} - \alpha_k|,$$

where $\alpha_k = \lim_{n\to\infty} \tilde{c}_{nk}$.

(c) *If* $A \in (c_0(\Theta), \ell_\infty)$, *then*

$$0 \le \|\mathcal{L}_A\|_\chi \le \limsup_{n\to\infty} \sum_k |\tilde{c}_{nk}|.$$

(d) *If* $A \in (c_0(\Theta), \ell_1)$, *then*

$$\lim_{s\to\infty} \|A\|^{[k]}_{(c_0(\Theta),\ell_1)} \le \|\mathcal{L}_A\|_\chi \le 4 \lim_{s\to\infty} \|A\|^{[k]}_{(c_0(\Theta),\ell_1)},$$

where

$$\|A\|^{[k]}_{(c_0(\Theta),\ell_1)} = \sup_{E\in E_0} \sum_k \left| \sum_{n\in E} \tilde{c}_{nk} \right|, k \in \mathbb{N}.$$

Proof. (a) Let $A \in (c_0(\Theta), c_0)$. It can be observed that

$$||A_n||^*_{c_0(\Theta)} = ||\mathfrak{A}_n||^*_{c_0} = ||\mathfrak{A}_n||_{\ell_1} = \sum_k |\tilde{c}_{nk}|$$

for $n \in \mathbb{N}$. Thus, using Part (a) of Lemma 6.5, we conclude that

$$\|\mathcal{L}_A\|_\chi = \limsup_{n\to\infty} \sum_k |\tilde{c}_{nk}|.$$

(b) Notice that

$$||\mathfrak{A}_n - \alpha_k||^*_{c_0} = ||\mathfrak{A}_n - \alpha_k||_{\ell_1} = \sum_k |\tilde{c}_{nk} - \alpha_k| \tag{21}$$

For each $n \in \mathbb{N}$, if $A \in (c_0(\Theta), c)$, then by Lemma 6.7, we obtain that $\mathfrak{A} \in (c_0, c)$. Using Part (b) of Lemma 6.5, we can infer that

$$\frac{1}{2}\limsup_{n\to\infty}||\mathfrak{A}_n - \alpha||^*_{c_0} \le \|\mathcal{L}_A\|_\chi \le \limsup_{n\to\infty}||\mathfrak{A}_n - \alpha||^*_{c_0},$$

Now, using the earlier expression (21), yields

$$\frac{1}{2}\limsup_{n\to\infty}\sum_k |\tilde{c}_{nk} - \alpha_k| \le \|\mathcal{L}_A\|_\chi \le \limsup_{n\to\infty}\sum_k |\tilde{c}_{nk} - \alpha_k|,$$

which is the required result.

(c) This part is proven similarly to Parts (a) and (b), except that we use Part (c) of Lemma 6.5 instead of Part (a).

(d) Note that

$$\left\|\sum_{n\in\mathbb{N}} \mathfrak{A}_n\right\|_{c_0} = \left\|\sum_{n\in\mathbb{N}} \mathfrak{A}_n\right\|_{\ell_1} = \sum_k \left|\sum_{n\in\mathbb{N}} \tilde{c}_{nk}\right|. \tag{22}$$

Assuming $A \in (c_0(\Theta), \ell_1)$, then by Lemma 6.7, we have $\mathfrak{A} \in (c_0, \ell_1)$. Consequently, by applying Lemma 6.6, we obtain

$$\lim_{k\to\infty}\left(\sup_{E\in E_0}\left\|\sum_{n\in E} \mathfrak{A}_n\right\|^*_{c_0}\right) \le \|\mathcal{L}_A\|_\chi \le 4\cdot\lim_{k\to\infty}\left(\sup_{E\in E_0}\left\|\sum_{n\in\mathbb{N}} \mathfrak{A}_n\right\|^*_{c_0}\right),$$

by using the earlier equation (22), reduces to

$$\lim_{k\to\infty}\|A\|^{[k]}_{(c_0(\Theta),\ell_1)} \le \|\mathcal{L}_A\|_\chi \le 4\lim_{k\to\infty}\|\Omega\|^{[k]}_{(c_0(\Theta),\ell_1)},$$

as desired.

Corollary 6.9. *The following assertions are true:*

(a) *Let* $A \in (c_0(\Theta), c_0)$, *then* $\mathcal{L}_A$ *is compact* $\Leftrightarrow$

$$\lim_{n\to\infty} \sum_k |\tilde{c}_{nk}| = 0.$$

(b) *Let* $A \in (c_0(\Theta), c)$, *then* $\mathcal{L}_A$ *is compact* $\Leftrightarrow$

$$\lim_{n\to\infty} \left(\sum_k |\tilde{c}_{nk} - \alpha_k| \right) = 0.$$

(c) *Let* $A \in (c_0(\Theta), \ell_\infty)$, *then* $\mathcal{L}_A$ *is compact if*

$$\lim_{n\to\infty} \sum_k |\tilde{c}_{nk}| = 0.$$

(d) *Let* $A \in (c_0(\Theta), \ell_1)$, *then* $\mathcal{L}_A$ *is compact* $\Leftrightarrow$

$$\lim_{k\to\infty} \left(\sup_{E\in E_0} \left(\sum_k \left| \sum_{n\in\mathbb{N}} \tilde{c}_{nk} \right| \right) \right) = 0.$$

7. Conclusion

Research on Pell-Lucas numbers has traditionally focused on fundamental aspects such as identities, recurrence relations, generating functions, Binet's formula, and special transformations, as well as their connections with hyperbolic quaternions (Horadam, 1994; Aydin, 2022). In this work, we extend the study by introducing the Pell-Lucas sequence space and conducting a detailed analysis of the spaces $c(\Theta)$ and $c_0(\Theta)$. The study further explores the fundamental properties and inclusion relationships of these sequence spaces, establishes a Schauder basis, and determines their α-, β-, and γ-duals. Our investigation also emphasizes the measure of non-compactness in $c_0(\Theta)$. We are hoping that the results presented here will be useful for future research in this area. Further studies may extend this work to the domain of the Pell-Lacus matrix in other sequence spaces such as Maddox's spaces, cs, and bs.

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