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Research Article

A new band matrix with *q*-Fibonacci

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Abstract: In this research paper, we define the Schröder basis of the space $\ell_p\left(\mathfrak{f}_q\right)$ defined by a new q-Fibonacci band matrix. We determine its dual spaces and matrix transformations. Finally, we prove that this space is of Banach-Saks type *p* and has the weak fixed point property.

Keywords: q-Fibonacci numbers; q-Fibonacci analogue; Dual spaces; Matrix transformation.

MSC 2020: 40G15, 40C05, 46A35, 42B08.

1. Introduction

Leonardo Fibonacci defined the Fibonacci numbers in 1202 as follows:

$$\mathfrak{f}_n = \mathfrak{f}_{n-1} + \mathfrak{f}_{n-2}$$

where $\mathfrak{f}_0 = 0$, $\mathfrak{f}_1 = 1$ and $n \geq 2$.

These numbers represent the golden ratio, which is derived from the ratio of two successive terms. They are used in a variety of fields, including mathematics.

In 1917, Schur (Schur, 1917) first defined the q-Fibonacci numbers as

$$\mathfrak{f}_n(q) = \mathfrak{f}_{n-1}(q) + q^{n-2}\mathfrak{f}_{n-2}(q)$$

where $\mathfrak{f}_0(q) = 0$, $\mathfrak{f}_1(q) = 1$ and $n \geq 2$. Let us write a few terms of the sequence

$$(\mathfrak{f}_n(q)) = (\mathfrak{f}_1(q), \mathfrak{f}_2(q), \mathfrak{f}_3(q), \mathfrak{f}_4(q), \dots)$$

explicitly as follows:

$$\mathfrak{f}_1(q) = 1,\ \mathfrak{f}_2(q) = \underbrace{\mathfrak{f}_1(q)}_{1} + q^0 \underbrace{\mathfrak{f}_0(q)}_{0} = 1,\ \mathfrak{f}_3(q) = \underbrace{\mathfrak{f}_2(q)}_{1} + q^1 \underbrace{\mathfrak{f}_1(q)}_{1} = 1 + q,$$

$$\mathfrak{f}_4(q) = \underbrace{\mathfrak{f}_3(q)}_{1+q} + q^2 \underbrace{\mathfrak{f}_2(q)}_{1} = 1 + q + q^2,$$

$$\mathfrak{f}_5(q) = \underbrace{\mathfrak{f}_4(q)}_{1+q+q^2} + q^3 \underbrace{\mathfrak{f}_3(q)}_{1+q} = 1 + q + q^2 + q^3 + q^4.$$

Carlitz (Carlitz, 1974) conducted significant research on these numbers, representing each q-Fibonacci number in the form of

$$\mathfrak{f}_{n+1}(q) = \sum_{2k \leq n} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q$$

with q-binomial coefficient. The works of (Andrews, 2004), (Hirschhorn, 1972), (Aytaç, 2018), (Koshy, 2001), and (Kac & Cheung, 2002) provide significant identities and research on these numbers.

Let us now present some notations associated with q-integers to facilitate the reader's clear understanding. The notation for q-integers, q-factorials, and q-binomial coefficients are denoted as follows:

$$[n]_q = \begin{cases} n, & q = 1 \\ \dfrac{1-q^n}{1-q}, & q \neq 1, \end{cases}$$

$$[n]_q! = [1]_q[2]_q \ldots [n]_q \qquad (n > 0), [0]_q! = 1,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q!\,[k]_q!},$$

respectively.

As $q \to 1$ approaches, the q-Fibonacci numbers return to the usual Fibonacci numbers.

Authors such as (Aktuğlu & Bekar, 2011), (Mursaleen et al., 2006), (Selmanogullari et al., 2011), (Yaying et al., 2021; 2022; 2023), (Dağlı, 2023), (Ellidokuzoglu et al. 2025), (Demiriz & Şahin, 2016), (Çınar & Et, 2020), (Atabey et al., 2023; 2025a; 2025b) have implemented q-numbers and special matrix in summability theory.

Let consider that ω is the collection of all real or complex number sequences, and ℓ_p is the Banach space of p-absolutely convergent sequences with the standard ℓ_p norm:

$$\|x\|_p = \left(\sum_k |x|^p\right)^{\frac{1}{p}}$$

for ($1 \le p < \infty$), and for $p = \infty$:

$$\|x\|_\infty = \sup_{k\in N} |x_k|,$$

the Banach spaces of bounded, convergent, and null convergent sequences $x = (x_k)$ are represented as ℓ_∞, c, and c_0, respectively.

It is presumed that $A = (a_{nk})$ is an infinite matrix that contains real or complex numbers.

$$Ax = \big(A_n(x)\big) = \sum_{k=1}^{\infty} a_{nk} x_k$$

is the definition of the A-transform of a sequence $x = (x_k)$. The series $\sum_{k=1}^{\infty} a_{nk} x_k$ must converge for every $n \in N$.

In the event that X and Y are two sequence spaces and $Ax \in Y$ for every sequence $x \in X$, the operator A establishes a matrix mapping from the spaces X to Y. The representation of (X, Y) is the collection of all matrices that perform mappings from set X to set Y.

A sequence space is the domain X_A that is associated with the matrix A within the space X. The definition is as follows:

$$X_A = \{x \in \omega : Ax \in X\}.$$

In addition, the sequence space X_A is classified as a BK-space if X is a BK-space and A represents a triangular matrix. This is defined by the norm $\|x\|_{X_A} = \|Ax\|_X$.

A Banach space X is said to possess the Banach-Saks property if every bounded sequence (x_n) in X permits a subsequence (z_n) such that the sequence $\{t_k(z)\}$ is convergent in the norm in X (see (Mursaleen et al., 2006)), where

$$t_k(z) = \frac{1}{k+1} \sum_{i=0}^{k} z_k, \quad k \in N.$$

It is said that a Banach space X has the weak Banach-Saks property if there is a subsequence (z_n) of (x_n) such that the sequence $\{t_k(z)\}$ is strongly convergent to 0, given any weakly null sequence $(x_n) \subset X$.

The coefficient provided by García-Falset in (Garcia, 1994) is as follows:

$$R(X) = \sup\left\{\lim_{n\to\infty} inf\|x_n - x\| : (x_n) \subset B(X), x_n \xrightarrow{w} x, x \in B(X)\right\},$$

where $B(X)$ represents X's unit ball.

Remark 1.1. *According to* Garcia (Garcia, 1997), *a Banach space X with $R(X) < 2$ possesses the weak fixed point property.*

Consider the range $1 < p < \infty$. A Banach space is characterized by having the Banach-Saks type p or the property (BS_p) if every weakly null sequence There exists a subsequence x_{k_l} of x_k such that for a certain constant $C > 0$,

$$\left\|\sum_{l=0}^{n} x_{k_l}\right\| < C(n+1)^{\frac{1}{p}}$$

for all $n \in N$ (see (Knaust, 1992)).

1.1. Preliminaries: New q-Fibonacci Band matrix

Atabey (2025) defined a new q-Fibonacci band matrix $\mathfrak{f}_q = (\mathfrak{f}_{nk}(q))$ by

$$\mathfrak{f}_{nk}(q) = \begin{cases} \dfrac{q^{n-2}\mathfrak{f}_{n-2}(q)}{\mathfrak{f}_n(q)}, & k = n-1 \\ \dfrac{\mathfrak{f}_{n-1}(q)}{\mathfrak{f}_n(q)}, & k = n \\ 0, & otherwise \end{cases}$$

where $\mathfrak{f}_n(q)$ be n th Fibonacci number, $n \in \{1,2,\ldots\}$ and $q \in R^+$.

Then, Atabey (2025) determined the $\mathfrak{f}_q$-transformation

$$y_n = \frac{\mathfrak{f}_{n-1}(q)}{\mathfrak{f}_n(q)} x_n + \frac{q^{n-2}\mathfrak{f}_{n-2}(q)}{\mathfrak{f}_n(q)} x_{n-1} \tag{1}$$

and its inverse transformation

$$x_n = \frac{1}{\mathfrak{f}_{n-1}(q)} \sum_{k=1}^{n} (-1)^{n+k} q^{\frac{n(n-1)-k(k-1)}{2}} \mathfrak{f}_k(q) y_k.$$

Atabey (2025) gave the definitions of the sequence spaces spaces $\ell_p(\mathfrak{f}_q)(1 \le p < \infty)$, $\ell_\infty(\mathfrak{f}_q)$, $c_0(\mathfrak{f}_q)$, and $c(\mathfrak{f}_q)$ as follows:

$$l_p(\mathfrak{f}_q) = \left\{x = (x_k) \in \omega : \sum_n \left|\frac{\mathfrak{f}_{n-1}(q)}{\mathfrak{f}_n(q)} x_n + \frac{q^{n-2}\mathfrak{f}_{n-2}(q)}{\mathfrak{f}_n(q)} x_{n-1}\right|^p < \infty\right\},$$

$$l_\infty(\mathfrak{f}_q) = \left\{x = (x_k) \in \omega: \sup_{n\in N}\left|\frac{\mathfrak{f}_{n-1}(q)}{\mathfrak{f}_n(q)}x_n + \frac{q^{n-2}\mathfrak{f}_{n-2}(q)}{\mathfrak{f}_n(q)}x_{n-1}\right| < \infty\right\},$$

$$c(\mathfrak{f}_q) = \left\{x = (x_k) \in \omega: \lim_{n\to\infty}(\mathfrak{f}_q)_n(x) \ exists\right\},$$

$$c_0(\mathfrak{f}_q) = \left\{x = (x_k) \in \omega: \lim_{n\to\infty}(\mathfrak{f}_q)_n(x) = 0\right\}.$$

Finally, Atabey (2025) showed that these spaces are BK-spaces, that the spaces $\ell_p(\mathfrak{f}_q)$, $c_0(\mathfrak{f}_q)$ and $c(\mathfrak{f}_q)$ are isomorphic to the well-known classical sequence spaces, and gave some inclusion relations.

2. Main Results

In this section, we define a Schröder basis for $\ell_p(\mathfrak{f}_q)$ and give the α-, β-, γ-duals, matrix transformations and some geometric properties of this space.

2.1. Basis

Theorem 2.1.1. *For every fixed $k \in N$ and $1 \leq p < \infty$, consider $¥^{(k)} \in \ell_p(\mathfrak{f}_q)$ defined as*

$$\left(¥^{(k)}\right)_n = \begin{cases} \dfrac{(-1)^{n+k}q^{\frac{n(n-1)-k(k-1)}{2}}\mathfrak{f}_k(q)}{\mathfrak{f}_{n-1}(q)}, & 1 \leq k \leq n \quad (n \in N). \\ 0, & k > n \end{cases} \tag{2}$$

Therefore, the sequence $\left(¥^{(k)}\right)_{k=1}^{\infty}$ serves as a basis for the space $\ell_p(\mathfrak{f}_q)$. Furthermore, for every element $x \in \ell_p(\mathfrak{f}_q)$, the expression

$$x = \sum_k (\mathfrak{f}_q)_k(x)¥^{(k)} \tag{3}$$

is unique.

Proof. Consider the $1 \leq p < \infty$. Subsequently, (2) indicates that $(\mathfrak{f}_q)\left(¥^{(k)}\right) = e^{(k)} \in \ell_p(\mathfrak{f}_q)$, which implies that $¥^{(k)} \in \ell_p(\mathfrak{f}_q)$.

Given $x \in \ell_p(\mathfrak{f}_q)$, let us set

$$x^m = \sum_k (\mathfrak{f}_q)_k(x)¥^{(k)}.$$

Hence, we may express that in writing

$$\mathfrak{f}_q(x^m) = \sum_{k=0}^{m} (\mathfrak{f}_q)_k(x)\,(\mathfrak{f}_q)(¥^{(k)}) = \sum_{k=0}^{m} (\mathfrak{f}_q)_k(x)e^{(k)}$$

and after

$$(\mathfrak{f}_q)_n(x - x^m) = \begin{cases} 0, & 0 \le n \le m \\ (\mathfrak{f}_q)_n(x), & n > m \end{cases} (n, m \in N).$$

Presently, there exists an $m_0 \in N$ for each given $\varepsilon > 0$ such that

$$\sum_{k=m_0+1}^{\infty} \left|(\mathfrak{f}_q)_n(x)\right|^p = \left(\frac{\varepsilon}{2}\right)^p.$$

Therefore, for every $m > m_0$, we get

$$\|x - x^m\|_{\ell_p(\mathfrak{f}_q)} = \left(\sum_{k=m+1}^{\infty} \left|(\mathfrak{f}_q)_n(x)\right|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=m_0+1}^{\infty} \left|(\mathfrak{f}_q)_n(x)\right|^p\right)^{\frac{1}{p}} \le \frac{\varepsilon}{2} \le \varepsilon,$$

$\lim_{m\to\infty} \|x - x^m\|_{\ell_p(\mathfrak{f}_q)} = 0$, proving that x is expressed as in (3).

Our hypothesis is that there is an additional form (3) similar to

$$x = \sum_k (l_q)_k(x)¥^{(k)}$$

that demonstrates the expression's uniqueness. The continuous transform T, whose isomorphism we have established in Theorem 2.2.6 of (Atabey, 2025), can be expressed as the following equation:

$$(\mathfrak{f}_q)_n(x) = \sum_k (l_q)_k(x)(\mathfrak{f}_q)_n(¥^{(k)}) = \sum_k (l_q)_k(x)\,\delta_{nk} = (l_q)_n(x).$$

This demonstrates that the form (2) is distinctive. Thus, the proof is concluded.

2.2. The duals of the $\ell_p(\mathfrak{f}_q)$ space

The α-, β-, and γ-duals of $\ell_p(\mathfrak{f}_q)$ are delineated in this section. Given that $p = 1$ can be demonstrated through analogy, we will investigate the scenario in which $1 < p \le \infty$. In order to substantiate Theorem 2.2.6 and Theorem 2.2.7, we refer to the lemmas in Stieglitz and Tietz (Stieglitz & Tietz, 1977).

Definition 2.2.1. *Consider that X is a sequence space. The α-, β-, and γ- duals*

$$X^\alpha = \{a = (a_k) \in \omega : ax = (a_k x_k) \in \ell_1, \forall x \in X\},$$

$$X^\beta = \{a = (a_k) \in \omega : ax = (a_k x_k) \in cs, \forall x \in X\},$$

$$X^{\gamma} = \{a = (a_k) \in \omega : ax = (a_k x_k) \in bs, \forall x \in X\},$$

defines X^{α}, X^{β}, and X^{γ} of a sequence space X, correspondingly.

F is the abbreviation for the family of all finite subsets of N. For $1 < p \leq \infty$, $\frac{1}{p} + \frac{1}{r} = 1$.

Readers are advised to consult the book by (Başar & Dutta, 2020) and the article by (Kara, 2013) in order to fully understand the studies discussed in this section.

Lemma 2.2.2. $A = (a_{nk}) \in (\ell_p, \ell_1)$ *iff* $(1 < p \leq \infty)$, $\left(\frac{1}{p} + \frac{1}{r} = 1\right)$

$$\sup_{K \in F} \sum_k \left| \sum_{n \in K} a_{nk} \right|^r < \varepsilon.$$

Lemma 2.2.3. $A = (a_{nk}) \in (\ell_p, c)$ *iff*

$$For\ (\forall\ k \in N)\ \lim_{n \to \infty} a_{nk}\ \ exists \tag{4}$$

$$\sup_{n \in N} \sum_k |a_{nk}|^r < \infty \quad \left(\frac{1}{p} + \frac{1}{r} = 1\right). \tag{5}$$

Lemma 2.2.4. $A = (a_{nk}) \in (\ell_\infty, c)$ *iff* (4) *holds and*

$$\lim_{n \to \infty} \sum_k |a_{nk}| = \sum_k \left| \lim_{n \to \infty} a_{nk} \right|. \tag{6}$$

Lemma 2.2.5. $A = (a_{nk}) \in (\ell_p, \ell_\infty)$ *iff* (5) *holds with* $(1 < p \leq \infty)$.

Theorem 2.2.6. *The set*

$$\mathrm{D}_1 = \left\{ a = (a_k) \in \omega : \sup_{K \in F} \sum_k \left| \sum_{n \in K} \frac{(-1)^{n+k} q^{\frac{n(n-1)-k(k-1)}{2}} \mathfrak{f}_k(q)}{\mathfrak{f}_{n-1}(q)} a_n \right|^r < \infty \right\}$$

is the α-dual of the $\ell_p(\mathfrak{f}_q)$ *space for* $1 < p \leq \infty$.

Proof. Consider $1 < p \leq \infty$, we define the matrix $B = (b_{nk})$ as

$$b_{nk}=\begin{cases}\dfrac{(-1)^{n+k}q^{\frac{n(n-1)-k(k-1)}{2}}\mathfrak{f}_k(q)}{\mathfrak{f}_{n-1}(q)}a_n, & 1\le k\le n\\ 0, & k>n\end{cases}$$

for any fixed sequence $a=(a_n)\in\omega$.

For every $x=(x_n)\in\omega$, we also assign $y=\mathfrak{f}_q x$. Subsequently, (1) follows.

$$a_nx_n=\sum_{k=1}^{n}\frac{(-1)^{n+k}q^{\frac{n(n-1)-k(k-1)}{2}}\mathfrak{f}_k(q)}{\mathfrak{f}_{n-1}(q)}a_ny_k=B_n(y)\ (n\in N). \tag{7}$$

As a consequence of (7), it is evident that $ax=(a_nx_n)\in\ell_1$ whenever $x\in\ell_p(\mathfrak{f}_q)$ iff $By\in\ell_1$whenever $y\in\ell_p$.

By employing Lemma 2.2.2, it is evident that

$$\sup_{K\in F}\sum_k\left|\sum_{n\in K}\frac{(-1)^{n+k}q^{\frac{n(n-1)-k(k-1)}{2}}\mathfrak{f}_k(q)}{\mathfrak{f}_{n-1}(q)}a_n\right|^r<\infty$$

and so $\left(\ell_p(\mathfrak{f}_q)\right)^\alpha=\mathfrak{D}_1$.

Theorem 2.2.7. *The sets* $\mathfrak{D}_2$, $\mathfrak{D}_3$ *and* $\mathfrak{D}_4$ *are defined as follows*:

$$\mathfrak{D}_2=\left\{a=(a_k)\in\omega:\sum_{j=k}^{\infty}\frac{(-1)^{j+k}q^{\frac{j(j-1)-k(k-1)}{2}}\mathfrak{f}_k(q)}{\mathfrak{f}_{j-1}(q)}a_j\ \ exists, \forall\, k\in N\right\},$$

$$\mathfrak{D}_3=\left\{a=(a_k)\in\omega:\sup_{n\in N}\sum_{k=1}^{n}\left|\sum_{j=k}^{n}\frac{(-1)^{j+k}q^{\frac{j(j-1)-k(k-1)}{2}}\mathfrak{f}_k(q)}{\mathfrak{f}_{j-1}(q)}a_j\right|^r<\infty\right\},$$

$$\mathfrak{D}_4=\left\{a=(a_k)\in\omega:\lim_{n\to\infty}\sum_{k=1}^{n}\left|\sum_{j=k}^{n}\frac{(-1)^{j+k}q^{\frac{j(j-1)-k(k-1)}{2}}\mathfrak{f}_k(q)}{\mathfrak{f}_{j-1}(q)}a_j\right|^r=\sum_k\left|\sum_{j=k}^{\infty}\frac{(-1)^{j+k}q^{\frac{j(j-1)-k(k-1)}{2}}\mathfrak{f}_k(q)}{\mathfrak{f}_{j-1}(q)}a_j\right|^r<\infty\right\}.$$

Following this, we have

I) $\left(\ell_p(\mathfrak{f}_q)\right)^\beta=\mathfrak{D}_2\cap\mathfrak{D}_3$ *for* $1<p\le\infty$,

II) $\left(\ell_\infty(\mathfrak{f}_q)\right)^\beta = \mathfrak{D}_2 \cap \mathfrak{D}_4$.

Proof. Assume that $a = (a_k) \in \omega$ and that $D = (d_{nk})$ is an

$$d_{nk} = \begin{cases} \sum_{j=k}^{n} \frac{(-1)^{j+k} q^{\frac{j(j-1)-k(k-1)}{2}} \mathfrak{f}_k(q)}{\mathfrak{f}_{j-1}(q)} a_j, & 1 \le k \le n \\ 0, & k > n \end{cases}$$

matrix.

Think about the following equality now.

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} a_k \left(\sum_{j=1}^{n} \frac{(-1)^{j+k} q^{\frac{j(j-1)-k(k-1)}{2}} \mathfrak{f}_k(q)}{\mathfrak{f}_{j-1}(q)} y_j \right)$$

$$= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} \frac{(-1)^{j+k} q^{\frac{j(j-1)-k(k-1)}{2}} \mathfrak{f}_k(q)}{\mathfrak{f}_{j-1}(q)} a_j \right) y_k = D_n(y) \tag{8}$$

Then, we can deduce from Lemma 2.2.3 that $ax = (a_k x_k) \in cs$ whenever $x \in \ell_p(\mathfrak{f}_q)$ iff $Dy \in c$ whenever $y = (y_k) \in \ell_p$. This is achieved by utilizing (1). Therefore, $(y_k) \in \left(\ell_p(\mathfrak{f}_q)\right)^\beta$ iff $(a_k) \in \mathfrak{D}_2$ and $(a_k) \in \mathfrak{D}_3$, as defined by (4) and (5), respectively. Consequently, $\left(\ell_p(\mathfrak{f}_q)\right)^\beta = \mathfrak{D}_2 \cap \mathfrak{D}_3$.

In the event that $p = \infty$, a comparable proof can be achieved by employing Lemma 2.2.4 in place of Lemma 2.2.3 and employing comparable methodologies.

Theorem 2.2.8. *For* $1 < p \le \infty$, $\left(\ell_p(\mathfrak{f}_q)\right)^\gamma = \mathfrak{D}_3$.

Proof. Lemma 2.2.5 provides the proof, which may be found using (7).

2.3. Matrix Transformations related to $\ell_p(\mathfrak{f}_q)$

The matrix classes $(\ell_p(\mathfrak{f}_q), X)$ for $1 < p \le \infty$, where $X \in \{\ell_\infty, \ell_1, c, c_0\}$, are characterized in this section. In order to maintain brevity, we employ

$$\dot{a}_{nk}=\sum_{j=k}^{\infty}\frac{(-1)^{j+k}q^{\frac{j(j-1)-k(k-1)}{2}}\mathfrak{f}_k(q)}{\mathfrak{f}_{j-1}(q)}a_{nj}.$$

We derive our findings in the subsequent lemma.

Lemma 2.3.1 (see [(Kirişçi & Başar, 2010), Theorem 4.1]**).** *Assume λ is a FK-space, U is a triangular matrix, V is its inverse, and μ is a random subset of ω. Define $C=(c_{nk})$ and $C^{(n)}=\left(c_{mk}^{(n)}\right)$*

$$c_{mk}^{(n)}=\begin{cases}\sum_{j=k}^{m}a_{nj}v_{jk}, & 1\le k\le m\\ 0, & k>m\end{cases},\ c_{nk}=\sum_{j=k}^{\infty}a_{nj}v_{jk},$$

respectively. Then, we have $A=(a_{nk})\in(\lambda_U,\mu)$ iff $C^{(n)}=\left(c_{mk}^{(n)}\right)\in(\lambda,c)$ and $C=(c_{nk})\in(\lambda,c)$.

Then we provide the prerequisites for valid results:

$$\sup_{m\in N}\sum_{k=1}^{m}\left|\sum_{j=k}^{m}\frac{(-1)^{j+k}q^{\frac{j(j-1)-k(k-1)}{2}}\mathfrak{f}_k(q)}{\mathfrak{f}_{j-1}(q)}a_j\right|^r<\infty,\tag{9}$$

$$\lim_{m\to\infty}\sum_{j=k}^{m}\frac{(-1)^{j+k}q^{\frac{j(j-1)-k(k-1)}{2}}\mathfrak{f}_k(q)}{\mathfrak{f}_{j-1}(q)}a_{nj}=\dot{a}_{nk}\ \ \forall\, n,k\in N,\tag{10}$$

$$\lim_{m\to\infty}\sum_{k=1}^{m}\left|\sum_{j=k}^{m}\frac{(-1)^{j+k}q^{\frac{j(j-1)-k(k-1)}{2}}\mathfrak{f}_k(q)}{\mathfrak{f}_{j-1}(q)}a_{nj}\right|=\sum_{k}|\dot{a}_{nk}|\ \ \forall\, n\in N,\tag{11}$$

$$\sup_{m\in N}\sum_{k}|\dot{a}_{nk}|^r<\infty,\tag{12}$$

$$\sup_{K\in F}\sum_{k}\left|\sum_{n\in N}\dot{a}_{nk}\right|^r<\infty,\tag{13}$$

$$\lim_{n\to\infty}\dot{a}_{nk}=\dot{\alpha}_{nk};\ \ k\in N,\tag{14}$$

$$\lim_{n\to\infty}\sum_k |\dot{a}_{nk}| = \sum_k |\dot{\alpha}_{nk}|;\ \ k \in N, \tag{15}$$

$$\lim_{n\to\infty}\sum_k \dot{a}_{nk} = 0, \tag{16}$$

$$\sup_{n,k\in N} |\dot{a}_{nk}| < \infty, \tag{17}$$

$$\sup_{k,m\in N}\left|\sum_{j=k}^{m}\frac{(-1)^{j+k}q^{\frac{j(j-1)-k(k-1)}{2}}\mathfrak{f}_k(q)}{\mathfrak{f}_{j-1}(q)}a_{nj}\right| < \infty, \tag{18}$$

$$\sup_{k\in N}\sum_n |\dot{a}_{nk}| < \infty, \tag{19}$$

$$\sup_{\mathrm{N},K\in F}\left|\sum_{n\in \mathrm{N}}\sum_{n\in K}\dot{a}_{nk}\right| < \infty. \tag{20}$$

Therefore, the results in (Stieglitz & Tietz, 1977), Lemma 2.3.1, and the aforementioned conditions can be used to derive the following.

Theorem 2.3.2.

I) $A = (a_{nk}) \in \left(\ell_1(\mathfrak{f}_q), \ell_\infty\right) \Leftrightarrow$ (10), (17) *and* (18) *hold.*

II) $A = (a_{nk}) \in \left(\ell_1(\mathfrak{f}_q), c\right) \Leftrightarrow$ (10), (14), (17) *and* (18) *hold.*

III) $A = (a_{nk}) \in \left(\ell_1(\mathfrak{f}_q), c_0\right) \Leftrightarrow$ (10), *with* $\dot{\alpha}_k = 0$, (14), (17) *and* (18) *hold.*

IV) $A = (a_{nk}) \in \left(\ell_1(\mathfrak{f}_q), \ell_1\right) \Leftrightarrow$ (10), (18) *and* (19) *hold.*

Theorem 2.3.3.

I) $A = (a_{nk}) \in \left(\ell_p(\mathfrak{f}_q), \ell_\infty\right) \Leftrightarrow$ (9), (10) *and* (12) *hold.*

II) $A = (a_{nk}) \in \left(\ell_p(\mathfrak{f}_q), c\right) \Leftrightarrow$ (9), (10), (12) *and* (14) *hold.*

III) $A = (a_{nk}) \in \left(\ell_p(\mathfrak{f}_q), c_0\right) \Leftrightarrow$ (9), (10), (12) *and with* $\dot{\alpha}_k = 0$ (14) *hold.*

IV) $A = (a_{nk}) \in (\ell_p(\mathfrak{f}_q), \ell_1) \Leftrightarrow$ (9), (10) *and* (13) *hold.*

Theorem 2.3.4.

I) $A = (a_{nk}) \in (\ell_\infty(\mathfrak{f}_q), \ell_\infty) \Leftrightarrow$ (10), (11) *and in case* $r = 1$ (12) *hold.*

II) $A = (a_{nk}) \in (\ell_\infty(\mathfrak{f}_q), c) \Leftrightarrow$ (10), (11), (14) *and* (15) *hold.*

III) $A = (a_{nk}) \in (\ell_\infty(\mathfrak{f}_q), c_0) \Leftrightarrow$ (10), (11) *and* (18) *hold.*

IV) $A = (a_{nk}) \in (\ell_\infty(\mathfrak{f}_q), \ell_1) \Leftrightarrow$ (10), (11) *and* (20) *hold.*

2.4. Some geometric properties of the $\ell_p(\mathfrak{f}_q)$

One of the most critical properties in Functional Analysis is the geometric properties of Banach spaces. For further information, please consult (Diestel, 2012), (Garcia, 1994; 1997), (Kananthai et al., 2002), (Knaust, 1992), (Mursaleen et al., 2006), (Savaş et al., 2009), (Hudzik et al., 2014).

We provide a list of geometric properties of the space $\ell_p(\mathfrak{f}_q)$ for $1 < p < \infty$ in this section.

The following conclusions can be derived from the geometric properties of the space $\ell_p(\mathfrak{f}_q)$, where $1 < p < \infty$.

Theorem 2.4.1. *The space* $\ell_p(\mathfrak{f}_q)$ *is of the Banach-Saks type p, with* $1 < p < \infty$.

Proof. A positive number sequence (ε_n) is such $\sum_{n=1}^{\infty} \varepsilon_n < \frac{1}{2}$ and let (x_n) be a weakly null sequence in $B\left(\ell_p(\mathfrak{f}_q)\right)$. Set $z_0 = x_0 = 0$ and $z_1 = x_{n_1} = x_1$. Thus, there is an $h_1 \in N$ such that

$$\left\| \sum_{i=h_1+1}^{\infty} z_1(i) e^{(i)} \right\|_{\ell_p(\mathfrak{f}_q)} < \varepsilon_1.$$

It is possible to find an $n_2 \in N$ such that

$$\left\| \sum_{i=0}^{h_1} x_n(i) e^{(i)} \right\|_{\ell_p(\mathfrak{f}_q)} < \varepsilon_1$$

$x_n \to 0$ coordinately when $n \geq n_2$, as (x_n) is a weakly null sequence. Set $z_2 = x_{n_2}$. After, there is an $h_2 > h_1$ such that

$$\left\| \sum_{i=h_2+1}^{\infty} z_2(i)e^{(i)} \right\|_{\ell_p(\mathfrak{f}_q)} < \varepsilon_2.$$

By utilizing the fact that $x_n \to 0$ coordinately, it is possible to determine an integer such that $n_3 > n_2$

$$\left\| \sum_{i=0}^{h_2} x_n(i)e^{(i)} \right\|_{\ell_p(\mathfrak{f}_q)} < \varepsilon_2,$$

when $n \geq n_3$.

We can identify two increasing subsequences h_i and n_i as we proceed through this process.

$$\left\| \sum_{i=0}^{h_j} x_n(i)e^{(i)} \right\|_{\ell_p(\mathfrak{f}_q)} < \varepsilon_j,$$

for each $n \geq n_{j+1}$ and

$$\left\| \sum_{i=h_j+1}^{\infty} z_n(i)e^{(i)} \right\|_{\ell_p(\mathfrak{f}_q)} < \varepsilon_j$$

where $z_j = x_{n_j}$. Thus,

$$\left\| \sum_{j=0}^{n} z_j \right\|_{\ell_p(\mathfrak{f}_q)} = \left\| \sum_{j=0}^{n} \left(\sum_{i=0}^{h_{j-1}} z_j(i)e^{(i)} + \sum_{i=h_{j-1}+1}^{h_j} z_j(i)e^{(i)} + \sum_{i=h_j}^{\infty} z_j(i)e^{(i)} \right) \right\|_{\ell_p(\mathfrak{f}_q)}$$

$$\leq \left\| \sum_{j=0}^{n} \left(\sum_{i=h_{j-1}+1}^{h_j} z_j(i)e^{(i)} \right) \right\|_{\ell_p(\mathfrak{f}_q)} + 2\sum_{i=0}^{n} \varepsilon_j.$$

In contrast, we observe that $\|x\|_{\ell_p(\mathfrak{f}_q)} \leq 1$. As a result, we have that

$$\left\|\sum_{j=0}^{n}\left(\sum_{i=h_{j-1}+1}^{h_j} z_j(i)e^{(i)}\right)\right\|_{\ell_p(\mathfrak{f}_q)}^{p} =$$

$$= \sum_{j=0}^{n}\sum_{i=h_{j-1}+1}^{h_j}\left|\frac{\mathfrak{f}_{i-1}(q)}{\mathfrak{f}_i(q)} z_j(i) + \frac{q^{i-2}\mathfrak{f}_{i-2}(q)}{\mathfrak{f}_i(q)} z_j(i-1)\right|^p$$

$$\leq \sum_{j=0}^{n}\sum_{i=0}^{\infty}\left|\frac{\mathfrak{f}_{i-1}(q)}{\mathfrak{f}_i(q)} z_j(i) + \frac{q^{i-2}\mathfrak{f}_{i-2}(q)}{\mathfrak{f}_i(q)} z_j(i-1)\right|^p \leq (n+1).$$

Thus, it is determined that

$$\left\|\sum_{j=0}^{n}\left(\sum_{i=h_{j-1}+1}^{h_j} z_j(i)e^{(i)}\right)\right\|_{\ell_p(\mathfrak{f}_q)} \leq (n+1)^{\frac{1}{p}}.$$

With the understanding that $1 \leq (n+1)^{\frac{1}{p}}$ for all $n \in N$ and $1 < p < \infty$, we can conclude that

$$\left\|\sum_{j=0}^{n} z_j\right\|_{\ell_p(\mathfrak{f}_q)} \leq (n+1)^{\frac{1}{p}} + 1 \leq 2(n+1)^{\frac{1}{p}}$$

is true. Consequently, $\ell_p(\mathfrak{f}_q)$ is of the Banach-Saks type p. Thus, the proof is concluded.

Remark 2.4.2. *Given that the $\ell_p(\mathfrak{f}_q)$ is linearly isomorphic to ℓ_p, $R\left(\ell_p(\mathfrak{f}_q)\right) = R(\ell_p) = 2^{\frac{1}{p}}$.*

We acquire the subsequent theorem as a consequence of Remarks 1.1 and 2.4.2.

Theorem 2.4.3. *The space $\ell_p(\mathfrak{f}_q)$ possesses the weak fixed-point property for $1 < p < \infty$.*

3. Conclusion

In this study, the Schröder basis of the space $\ell_p(\mathfrak{f}_q)$, which Atabey (Atabey, 2025) constructed with a new q-Fibonacci band matrix, was constructed. Furthermore, by computing dual spaces and matrix transformations, it was shown that this space is of the Banach-Saks type p and has the weak fixed-point property.

In the future, duals and matrix transformations of the spaces $c(\mathfrak{f}_q)$ and $c_0(\mathfrak{f}_q)$ can be studied.

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