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Research Article

# New Matrix Domains Arising from the Euler Totient Function and Its Summatory Function

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**Abstract:** The matrix formed using the Euler totient function together with its summatory function is employed to generate new sequence spaces. After establishing several features of these spaces, their duals are explicitly identified. Subsequent sections provide the characterization of a number of matrix transformations acting on the new domains. The paper concludes with an investigation of the compactness of operators associated with the aforementioned matrix on the space of null sequences.

**Keywords:**Arithmetic divisor sum function, Matrix domain, Dual space, Matrix mapping.

**MSC 2020:** 40C05, 46A45, 47B37, 47B07.

## 1. Introduction

In recent years, sequence spaces and summability methods have attracted considerable interest due to their broad applications in analysis. Using matrix domains associated with various summability matrices, many new sequence spaces have been introduced; see (Altay & Başar, 2006; Başar & Altay, 2003; Yaying & Hazarika, 2020, 2021; Alp & İlkhan, 2019; Alp, 2021; Alp & Kara, 2021; İlkhan & Kara, 2019; Demiriz et al., 2020; Demiriz & Erdem, 2020, 2023; Alp, 2025). Since any infinite matrix can be viewed as a linear operator between sequence spaces, the study of such transformations has remained central in summability theory; see (Başar, 2012; Malkowsky, 1997; Mursaleen & Noman, 2010a; Alp & Kara, 2018; Alp, 2023; Dağlı, 2022; Dağlı & Yaying, 2023; Gökçe, 2023; Kara & Bayrakdar, 2021; Devletli & Kara, 2023). Moreover, compact operators and the Hausdorff measure of noncompactness have been widely investigated in this context; see (Başarır & Kara, 2012a, 2012b; Kara & Başarır, 2011; Mursaleen & Noman, 2011; Mursaleen & Roopaei, 2021).

In some recent papers, the authors (Yaying & Saikia, 2022; Yaying et al., 2024) have constructed newmatrices by the aid of arithmetic divisor sumfunctions to introduce matrix domains of them in classical sequence spaces. Later, in (Kara & Aydın, 2025), a new band matrix has been defined using the Euler totient function together with its summatory function and new sequence spaces has been obtained by the aid of this matrix.

This study forms the starting point of the present study. In this framework, some sequence spaces are constructed as the domains of the newly defined band matrix in the spaces of convergent and null sequences. The paper further identifies the dual spaces of these new spaces and characterizes the associated matrix transformations.

The final section presents bounds for the Hausdorff measure of noncompactness of bounded linear operators acting on one of the resulting space.

## 2. Preliminaries

This section provides the fundamental definitions and preliminary results needed in the subsequent sections.

A sequence space is a linear subspace of the space $\omega$ of all sequences. The space of all finitely non-zero sequences $\omega_0$, the space of all bounded sequences $\ell_\infty$, the space of all convergent sequences $c$, the space of all null sequences $c_0$ and the space of all absolutely $p$- summable sequences $\ell_p$ are the examples for the classical sequence spaces. The spaces $c_0$, $c$ and $\ell_\infty$ are complete normed spaces with$\|x\|_{\ell_\infty} = \|x\|_c = \|x\|_{c_0} = \sup_{i\in\mathbb{N}}|x_i|$and the space $\ell_p$ is a complete normed space with$\|x\|_{\ell_p} = (\sum_i |x_i|^p)^{1/p}$,where $\mathbb{N} = \{1,2,3,\dots\}$. Unless stated otherwise, assume that $1 < p < \infty$ and $q = \frac{p}{p-1}$ is the conjugate of $p$.

A linear topological sequence space $\mathcal{X}$ is called a K-space provided that each functional $p_m: \mathcal{X} \to \mathbb{K}$, $p_m(x) = x_m$ is continuous for all $m \in \mathbb{N}$, where $\mathbb{K}$ is real or complex field. If a K-space $\mathcal{X}$ is a complete linear metric space, then it is called an FK-space. If the topology of an FK-space is normable, then it is called a BK-space. Let $e = (e_r)$ be the sequence with term $e_r = 1$ for all $r$ and $e^{(i)} = (e_r^{(i)})(i \in \mathbb{N})$ be the sequence with terms 1 if $i = r$ and 0 if $i \neq r$. Given any FK-space $\mathcal{X} \supset \omega_0$ and a sequence $x = (x_r)$ in $\mathcal{X}$, it is said that the sequence $x$ satisfies the AK-property if $(x^{[i]})$ converges to $x$, where $x^{[i]} = \sum_{r=1}^{i} x_r e^{(r)}$.

Let $\mathcal{A} = (a_{ij})$ be an infinite matrix and $\mathcal{A}_i$ be the sequence in the $i$th row of $\mathcal{A}$. The $\mathcal{A}$-transform of a sequence $x = (x_i) \in \omega$ is the sequence $\mathcal{A}x$ obtained by the usual matrix product and its terms are written as

$$\mathcal{A}_i(x) = \sum_j a_{ij}x_j$$

provided that the series is convergent for each $i \in \mathbb{N}$. If the sequence $\mathcal{A}x$ exists and $\mathcal{A}x \in \mathcal{Y}$ for all $x \in \mathcal{X}$, then $\mathcal{A}$ is called a matrix mapping from the sequence space $\mathcal{X}$ into the sequence space $\mathcal{Y}$. $(\mathcal{X}, \mathcal{Y})$ denotes the class of all infinite matrices from $\mathcal{X}$ into $\mathcal{Y}$.

The sequence space $\mathcal{X}_{\mathcal{A}}$ called the (matrix) domain of $\mathcal{A}$ in the space $\mathcal{X}$ is the set

$$\mathcal{X}_{\mathcal{A}} = \{x \in \omega : \mathcal{A}x \in \mathcal{X}\}.$$

A sequence $(b_i)$ in a normed space $(\mathcal{X}, \|.\|)$ is called a Schauder basis if for any $x \in \mathcal{X}$, there exists a unique scalar sequence $(a_i)$ satisfying$\|x - a_1b_1 + a_2b_2 + \ldots + a_ib_i\| \longrightarrow 0$ as $i \to \infty$. Then, $x = \sum_i a_ib_i$ holds.

**Theorem 2.1.** (Jarrah & Malkowsky, 2003) *Given any triangle $\mathcal{A}$ and its inverse $\tilde{\mathcal{A}}$, if a normed space $\mathcal{X}$ has a Schauder basis $\{b^{(i)}\}_{i\in\mathbb{N}}$, then $\{\tilde{\mathcal{A}}(b^{(i)})\}_{i\in\mathbb{N}}$ is a Schauder basis of the matrix domain $\mathcal{X}_{\mathcal{A}}$.*

The following result provides a characterization of certain classes of matrices, $\mathcal{N}$ is used to denote the family of all finite subsets of $\mathbb{N}$.

**Lemma 2.2.** (Stieglitz & Tietz, 1977)

| From/to | $\ell_\infty$ | $c$ | $c_0$ | $\ell_p$ | $\ell_1$ |
|---|---|---|---|---|---|
| $\ell_\infty$ | 1. | 4. | 9. | 14. | 16. |
| $c$ | 1. | 5. | 10. | 14. | 16. |
| $c_0$ | 1. | 6. | 11. | 14. | 16. |
| $\ell_p$ | 2. | 7. | 12. | – | 17. |
| $\ell_1$ | 3. | 8. | 13. | 15 | 18. |

1. $\mathcal{A} = (a_{ij}) \in (\ell_\infty, \ell_\infty) = (c, \ell_\infty) = (c_0, \ell_\infty) \Leftrightarrow$

$$\sup_i \sum_j |a_{ij}| < \infty \tag{1}$$

*holds.*

2. $\mathcal{A} = (a_{ij}) \in (\ell_p, \ell_\infty) \Leftrightarrow$

$$\sup_i \sum_j |a_{ij}|^q < \infty \tag{2}$$

*holds.*

3. $\mathcal{A} = (a_{ij}) \in (\ell_1, \ell_\infty) \Leftrightarrow$

$$\sup_{i,j} |a_{ij}| < \infty \tag{3}$$

*holds.*

4. $\mathcal{A} = (a_{ij}) \in (\ell_\infty, c) \Leftrightarrow$

$$\lim_i a_{ij} \text{ exists for all } j \in \mathbb{N} \tag{4}$$

*and*

$$\lim_i \sum_j |a_{ij}| = \sum_j \left|\lim_i a_{ij}\right|$$

*hold.*

5. $\mathcal{A} = (a_{ij}) \in (c, c) \Leftrightarrow$(1), (4) *hold and* $\lim_i \sum_j a_{ij}$ *exists.*

6. $\mathcal{A} = (a_{ij}) \in (c_0, c) \Leftrightarrow$ (1) *and* (4) *hold.*$\mathcal{A} = (a_{ij}) \in (\ell_p, c) \Leftrightarrow$(2) *and* (4) *hold.*

7. $\mathcal{A} = (a_{ij}) \in (\ell_1, c) \Leftrightarrow$(3) *and* (4) *hold.*

8. $\mathcal{A} = (a_{ij}) \in (\ell_\infty, c_0) \Leftrightarrow \lim_i \sum_j |a_{ij}| = 0$ *holds.*

9. $\mathcal{A} = (a_{ij}) \in (c, c_0) \Leftrightarrow$(1) *and*

$$\lim_i a_{ij} = 0, \text{ for all } j \in \mathbb{N} \tag{5}$$

*and*

$$\lim_i \sum_j a_{ij} = 0$$

*hold.*

10. $\mathcal{A} = (a_{ij}) \in (c_0, c_0) \Leftrightarrow$ (1) *and* (5) *hold.*

11. $\mathcal{A} = (a_{ij}) \in (\ell_p, c_0) \Leftrightarrow$(2) *and* (5) *hold.*

12. $\mathcal{A} = (a_{ij}) \in (\ell_1, c_0) \Leftrightarrow$(3) *and* (5) *hold.*

13. $\mathcal{A} = (a_{ij}) \in (\ell_\infty, \ell_p) = (c, \ell_p) = (c_0, \ell_p) \Leftrightarrow$

$$\sup_{K \in \mathcal{N}} \sum_i \left|\sum_{j \in K} a_{ij}\right|^p < \infty$$

*holds.*

14. $\mathcal{A} = (a_{ij}) \in (\ell_1, \ell_p) \Leftrightarrow$

$$\sup_j \sum_i |a_{ij}|^p < \infty$$

*holds.*

15. $\mathcal{A} = (a_{ij}) \in (\ell_\infty, \ell_1) = (c, \ell_1) = (c_0, \ell_1) \Leftrightarrow$

$$\sup_{N,K \in \mathcal{N}} \left| \sum_{i \in N} \sum_{j \in K} a_{ij} \right| < \infty \Leftrightarrow \sup_{N \in \mathcal{N}} \sum_j \left| \sum_{i \in N} a_{ij} \right| < \infty \Leftrightarrow \sup_{K \in \mathcal{N}} \sum_i \left| \sum_{j \in K} a_{ij} \right| < \infty.$$

16. $\mathcal{A} = (a_{ij}) \in (\ell_p, \ell_1) \Leftrightarrow$

$$\sup_{N \in \mathcal{N}} \sum_j \left| \sum_{i \in N} a_{ij} \right|^q < \infty$$

*holds.*

17. $\mathcal{A} = (a_{ij}) \in (\ell_1, \ell_1) \Leftrightarrow \sup_j \sum_i |a_{ij}| < \infty$*holds.*

The multiplier space of the sequence spaces $\mathcal{X}$ and $\mathcal{Y}$ consists of sequences $a \in \omega$ such that $ax \in \mathcal{Y}$ for any $x \in \mathcal{X}$ and it is denoted by $\mathcal{M}(\mathcal{X}, \mathcal{Y})$.If $\mathcal{Y} = \ell_1$, $\mathcal{Y} = cs$ or $\mathcal{Y} = bs$, the multiplier spaces$\mathcal{M}(\mathcal{X}, \ell_1)$, $\mathcal{M}(\mathcal{X}, cs)$and $\mathcal{M}(\mathcal{X}, bs)$ are called as$\alpha$-, $\beta$- and $\gamma$-duals of the space $\mathcal{X}$ and denoted by $\mathcal{X}^\alpha$, $\mathcal{X}^\beta$ and $\mathcal{X}^\gamma$, respectively.

The following theorem is essential to obtain the $\beta$- and $\gamma$-duals.

**Theorem 2.3.** (Altay & Başar, 2007) *Let $\mathcal{B} = (b_{ij})$ be defined via a sequence $a = (a_i) \in \omega$ and the inverse $\tilde{\mathcal{A}} = (\tilde{a}_{ij})$ of the triangle matrix $\mathcal{A} = (a_{ij})$ by*

$$b_{ij} = \begin{cases} \sum_{l=j}^{i} a_l \tilde{a}_{lj} & , \quad \text{if } 1 \le j \le i \\ 0 & , \quad \text{if } j > i. \end{cases}$$

*Then,*

$$\mathcal{X}_{\mathcal{A}}^{\beta} = \{a = (a_i) \in \omega : \mathcal{B} \in (\mathcal{X}, c)\},$$

*and*

$$\mathcal{X}_{\mathcal{A}}^{\gamma} = \{a = (a_i) \in \omega : \mathcal{B} \in (\mathcal{X}, \ell_\infty)\}.$$

Let $\tilde{\mathcal{X}}$ be a bounded set in a metric space $\mathcal{X}$, then Hausdorff measure of noncompactness $\chi(\tilde{\mathcal{X}})$ is defined by

$$\chi(\tilde{\mathcal{X}}) = \inf\{\varepsilon > 0 : \tilde{\mathcal{X}} \subseteq \cup_{i=1}^{j} B(x_i, r_i), x_i \in \mathcal{X}, r_i < \varepsilon, j \in \mathbb{N}\},$$

where $B(x_i, r_i)$ is the open ball centered at $x_i$ and radius $r_i$ for all $i = 1,2,\ldots,j$.

A linear operator $\mathcal{T}: \mathcal{X} \to \mathcal{Y}$ is said to be compact, if for every bounded sequence $x = (x_i)$ in $\mathcal{X}$, the sequence $(\mathcal{T}(x_i))$ possesses a convergent subsequence in $\mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. The notation $\|\mathcal{T}\|_\chi$ stands for the Hausdorff measure of noncompactness of $\mathcal{T}$, which is defined as

$$\|\mathcal{T}\|_\chi = \chi(\mathcal{T}(\{x \in \mathcal{X} : \|x\| = 1\})).$$

The connection between compact operators and the Hausdorff measure of noncompactness lies in the fact that anoperator$\mathcal{T}$ is compact if and only if its Hausdorff measure of noncompactness satisfies $\|\mathcal{T}\|_\chi = 0$.

**Definition 2.4.** (Malkowsky & Rakočević, 1998) *Given any BK-space $\mathcal{X} \supseteq \omega_0$ and any sequence $a = (a_j) \in \omega$, the value*

$$\|a\|_{\mathcal{X}}^* = \sup\left\{\left|\sum_j a_j x_j\right| : \|x\| = 1\right\}$$

*is well defined if $\sum_j a_j x_j$ is convergent for all $x \in \mathcal{X}$. If $a \in \mathcal{X}^\beta$, then it is clear that $\sum_j a_j x_j$ exists and is finite for all $x \in \mathcal{X}$.*

**Lemma 2.5.** (Malafosse, 2005) *Let $\mathcal{X}$ be one of the spaces $\ell_\infty$, $c$ or $c_0$. Then $\mathcal{X}^\beta = \ell_1$ and $\|a\|_{\mathcal{X}}^* = \sum_j |a_j|$.*

**Lemma 2.6.** (Malkowsky & Rakočević, 2000) *Given any BK-spaces $\mathcal{X}, \mathcal{Y}$ and any $\mathcal{A} \in (\mathcal{X}, \mathcal{Y})$, there exists a bounded linear operator $\mathcal{T}_\mathcal{A}$ such that $\mathcal{T}_\mathcal{A}(x) = \mathcal{A}x$ for all $x \in \mathcal{X}$.*

**Lemma 2.7.** (Malkowsky & Rakočević, 2000) *Given any BK-space $\mathcal{X} \supseteq \omega_0$ and $\mathcal{A} \in (\mathcal{X}, \mathcal{Y})$, the statement*

$$\|\mathcal{T}_\mathcal{A}\| = \|\mathcal{A}\|_{(\mathcal{X},\mathcal{Y})} = \sup_{i\in\mathbb{N}} \|\mathcal{A}_i\|_{\mathcal{X}}^* < \infty$$

*holds, where $\mathcal{Y}$ is one of the spaces $\ell_\infty$, $c$ or $c_0$.*

In order to estimate the Hausdorff measure of noncompactness on $c_0$, the following result is used.

**Theorem 2.8.** (Rakočević, 1998) *Given any bounded set $\tilde{\mathcal{X}}$ in $c_0$ and each operator $P_m: c_0 \to c_0 (m \in \mathbb{N})$ with $P_m(x) = (x_1, x_2, \ldots, x_m, 0,0,\ldots)$,*

$$\chi(\tilde{\mathcal{X}}) = \lim_{m\to\infty}\left(\sup_{x\in\tilde{\mathcal{X}}} \|(I - P_m)(x)\|_\infty\right)$$

*holds, where $I$ is the identity operator on $c_0$.*

**Lemma 2.9.** (Mursaleen & Noman, 2010b) *Given any BK-space $\mathcal{X} \supseteq \omega_0$, the following results hold.*

*(a) If* $\mathcal{A} \in (\mathcal{X}, \ell_\infty)$*, then* $0 \leq \|\mathcal{T}_\mathcal{A}\|_\chi \leq \limsup_i \|\mathcal{A}_i\|_\mathcal{X}^*$ *and if* $\lim_i \|\mathcal{A}_i\|_\mathcal{X}^* = 0$*, then* $\mathcal{T}_\mathcal{A}$ *is compact.*

*(b) If* $\mathcal{A} \in (\mathcal{X}, c_0)$*, then* $\|\mathcal{T}_\mathcal{A}\|_\chi = \limsup_i \|\mathcal{A}_i\|_\mathcal{X}^*$ *and* $\mathcal{T}_\mathcal{A}$ *is compact if and only if* $\lim_i \|\mathcal{A}_i\|_\mathcal{X}^* = 0$.

*(c) If* $\mathcal{X}$ *has AK or* $\mathcal{X} = \ell_\infty$*, and* $\mathcal{A} \in (\mathcal{X}, c)$*, then*

$$\frac{1}{2}\limsup_i \|\mathcal{A}_i - a\|_\mathcal{X}^* \leq \|\mathcal{T}_\mathcal{A}\|_\chi \leq \limsup_i \|\mathcal{A}_i - a\|_\mathcal{X}^*$$

*and* $\mathcal{T}_\mathcal{A}$ *is compact if and only if* $\lim_i \|\mathcal{A}_i - a\|_\mathcal{X}^* = 0$*,where* $a = (a_j)$ *and* $a_j = \lim_i a_{ij}$ *for all* $j \in \mathbb{N}$.

**Lemma 2.10.** (Mursaleen & Noman, 2010b) *Given any BK-space* $\mathcal{X} \supseteq \omega_0$*, if* $\mathcal{A} \in (\mathcal{X}, \ell_1)$*, then*

$$\lim_m \left( \sup_{N \in \mathcal{N}_m} \left\| \sum_{i \in N} \mathcal{A}_i \right\|_\mathcal{X}^* \right) \leq \|\mathcal{T}_\mathcal{A}\|_\chi \leq 4 \lim_m \left( \sup_{N \in \mathcal{N}_m} \left\| \sum_{i \in N} \mathcal{A}_i \right\|_\mathcal{X}^* \right)$$

*and* $\mathcal{T}_\mathcal{A}$ *is compact if and only if* $\lim_m \left( \sup_{N \in \mathcal{N}_m} \| \sum_{i \in N} \mathcal{A}_i \|_\mathcal{X}^* \right) = 0$*,where* $\mathcal{N}_m$ *denotes the subcollection of* $\mathcal{N}$ *consisting of all subsets of* $\mathbb{N}$ *whose elements are greater than* $m$.

The Euler totient function $\varphi$ gives the number of positive integers less than $i$ that are coprime to $i$. The function $\top$ defined by $\top(i) = \sum_{j=1}^{i} \varphi(j)$ denotes the Euler totient summation function. It gives the number of coprime integer pairs $p_1, p_2$ with $1 \leq p_1 \leq p_2 \leq i$.

In (Kara & Aydın, 2025), the authors have defined the matrix $\Delta(\varphi, \top) = (\delta(\varphi, \top)_{ij})$ as

$$\delta(\varphi, \top)_{ij} = \begin{cases} \dfrac{(-1)^{i-j}\top(j)}{\varphi(i)} & , \quad i-1 \leq j \leq i \\ 0 & , \quad \text{otherwise} \end{cases}$$

The inverse $\Delta(\varphi, \top)^{-1} = (\delta(\varphi, \top)_{ij}^{-1})$ has calculated as

$$\delta(\varphi, \top)_{ij}^{-1} = \begin{cases} \dfrac{\varphi(j)}{\top(i)} & , \quad \text{if } 1 \leq j \leq i \\ 0 & , \quad \text{if } j > i \end{cases}$$

Also, the authors have introduced the spaces

$$\ell_p(\Delta(\varphi, \top)) = \{x = (x_i) \in \omega : \sum_i \Big| \sum_{j=i-1}^{i} (-1)^{(i-j)} \frac{\top(j)}{\varphi(i)} x_j \Big|^p < \infty\} (1 \leq p < \infty)$$

and

$$\ell_\infty(\Delta(\varphi,\top)) = \{x=(x_i)\in\omega: \sup_i\left|\sum_{j=i-1}^{i}(-1)^{(i-j)}\frac{\top(j)}{\varphi(i)}x_j\right|<\infty\}.$$

## 3. Banach spaces $c_0(\Delta(\varphi,\top))$ and $c(\Delta(\varphi,\top))$

In this section, new sequence spaces $c_0(\Delta(\varphi,\top))$ and $c(\Delta(\varphi,\top))$ consisting of sequences whose $\Delta(\varphi,\top)$-transforms are in the spaces $c_0$ and $c$ are introduced. The $\Delta(\varphi,\top)$-transform of a sequence $x=(x_i)\in\omega$ is the sequence $y=(y_i)$ with

$$y_i=\Delta(\varphi,\top)_i(x)=\sum_{j=i-1}^{i}(-1)^{(i-j)}\frac{\top(j)}{\varphi(i)}x_j$$

for all $i\in\mathbb{N}$. Hence the spaces are

$$c_0(\Delta(\varphi,\top)) = \{x=(x_i)\in\omega: \lim_{i\to\infty}\sum_{j=i-1}^{i}(-1)^{(i-j)}\frac{\top(j)}{\varphi(i)}x_j=0\}$$

and

$$c(\Delta(\varphi,\top)) = \{x=(x_i)\in\omega: \lim_{i\to\infty}\sum_{j=i-1}^{i}(-1)^{(i-j)}\frac{\top(j)}{\varphi(i)}x_j \text{ exists}\}.$$

**Theorem 3.1.** *The spaces $c_0(\Delta(\varphi,\top))$ and $c(\Delta(\varphi,\top))$ are Banach sequence spaces with the norm $\|.\|_{\Delta(\varphi,\top)}$ defined by $\|x\|_{\Delta(\varphi,\top)}=\sup_{i\in\mathbb{N}}\left|\sum_{j=i-1}^{i}(-1)^{(i-j)}\frac{\top(j)}{\varphi(i)}x_j\right|$.*

*Proof.* One can easily prove that $\|.\|_{\Delta(\varphi,\top)}$ is a norm on the spaces $c_0(\Delta(\varphi,\top))$ and $c(\Delta(\varphi,\top))$.Given anyCauchy sequence $(x^i)$ in $c_0(\Delta(\varphi,\top))$ or $c(\Delta(\varphi,\top))$, the equality

$$\begin{aligned}\|x^i-x^m\|_{\Delta(\varphi,\top)} &= \|\Delta(\varphi,\top)(x^i-x^m)\|_{\ell_\infty}\\ &= \|\Delta(\varphi,\top)x^i-\Delta(\varphi,\top)x^m\|_{\ell_\infty}=\|y^i-y^m\|_{\ell_\infty}\end{aligned}$$

means that $(y^i)$ is a Cauchy sequence in $c_0$ or $c$. Since $c_0$ and $c$ are Banach spaces, this Cauchy sequenceconverges to a sequence $y$ in $c_0$ or $c$. Set $x=\Delta(\varphi,\top)^{-1}y$. Then

$$\begin{aligned}\lim_{i\to\infty}\|x^i-x\|_{\Delta(\varphi,\top)} &= \lim_{i\to\infty}\|\Delta(\varphi,\top)(x^i-x)\|_{\ell_\infty}\\ &= \lim_{i\to\infty}\|\Delta(\varphi,\top)x^i-\Delta(\varphi,\top)x\|_{\ell_\infty}=\lim_{i\to\infty}\|y^i-y\|_{\ell_\infty}=0\end{aligned}$$

holds which yields $\lim_{i\to\infty}x^i=x$. This completes the proof.

**Remark 3.2.** *The spaces $c_0(\Delta(\varphi,\top))$ and $c(\Delta(\varphi,\top))$ are BK-spaces.*

**Theorem 3.3.** *The spaces* $c_0(\Delta(\varphi,\top))$ *and* $c(\Delta(\varphi,\top))$ *are linearly isomorphic to* $c_0$ *and* $c$*, respectively.*

*Proof*. The proof follows with the same way at the proof of Theorem 4.3 in (Kara & Aydın, 2025) by using the mapping $T: c_0(\Delta(\varphi,\top)) \to c_0$ or $T: c(\Delta(\varphi,\top)) \to c$, $T(u) = \Delta(\varphi,\top)u$.

**Theorem 3.4.** *The inclusion*

$$c_0(\Delta(\varphi,\top)) \subset c(\Delta(\varphi,\top)) \subset \ell_\infty(\Delta(\varphi,\top))$$

*Proof*. Let $x \in c_0(\Delta(\varphi,\top))$. Then, $\Delta(\varphi,\top)x \in c_0$ and so $\Delta(\varphi,\top)x \in c$ which implies that $x \in c(\Delta(\varphi,\top))$. In the same manner, $c(\Delta(\varphi,\top)) \subset \ell_\infty(\Delta(\varphi,\top))$ holds.

Choose a sequence $y = (y_i) \in c \setminus c_0$ or $y = (y_i) \in \ell_\infty \setminus c$. Hence, the sequence $x = (x_i) \in c(\Delta(\varphi,\top)) \setminus c_0(\Delta(\varphi,\top))$ or $x = (x_i) \in \ell_\infty(\Delta(\varphi,\top)) \setminus c(\Delta(\varphi,\top))$, where

$$x_i = \sum_{j=1}^{i} \frac{\varphi(j)}{\top(i)} y_j$$

for all $i \in \mathbb{N}$.

**Theorem 3.5.** *Consider the sequence* $b^{(j)} = (b_i^{(j)})$ *defined by*

$$b_i^{(j)} = \begin{cases} \dfrac{\varphi(j)}{\top(i)} & , \quad \text{if } 1 \le j \le i \\ 0 & , \quad \text{if } j > i, \end{cases}$$

*where* $b^{(j)} \in c_0(\Delta(\varphi,\top))$ *for each* $j \in \mathbb{N}$*. Then, the following statements hold:*

*Proof*. The proof follows from Theorem 2.3 of (Jarrah & Malkowsky, 2003).

## 4. The $\alpha$-, $\beta$- and $\gamma$-duals of the spaces $c_0(\Delta(\varphi,\top))$ and $c(\Delta(\varphi,\top))$

Firstly, the $\alpha$-dual of the spaces $c_0(\Delta(\varphi,\top))$ and $c(\Delta(\varphi,\top))$ is established.

**Theorem 4.1.** *The* $\alpha$*-dual of the spaces* $c_0(\Delta(\varphi,\top))$ *and* $c(\Delta(\varphi,\top))$ *is the set*

$$A = \{a = (a_i) \in \omega : \sup_{N,M \in \mathcal{N}} \left| \sum_{i \in N} \sum_{j \in M} \frac{\varphi(j)}{\top(i)} a_i \right| < \infty\}.$$

*Proof*. Let $a = (a_i) \in \omega$ and $\mathcal{A} = (a_{ij})$ be an infinite matrix with terms

$$a_{ij} = \begin{cases} \dfrac{\varphi(j)}{\top(i)} a_i & , \quad \text{if } 1 \le j \le i \\ 0 & , \quad \text{if } j > i. \end{cases}$$

Hence, it is obtained that

$$a_i x_i = \sum_{j=1}^{i} \frac{\varphi(j)}{\top(i)} a_i y_j = \mathcal{A}_i y$$

for any $x = (x_i) \in c_0(\Delta(\varphi, \top))$ or $x = (x_i) \in c(\Delta(\varphi, \top))$. It follows that $ax \in \ell_1$ for $x \in c_0(\Delta(\varphi, \top))$ or $x \in c(\Delta(\varphi, \top))$ if and only if $\mathcal{A}y \in \ell_1$ for $y \in c_0$ or $y \in c$. That is, $a \in (c_0(\Delta(\varphi, \top)))^\alpha$ or $a \in (c(\Delta(\varphi, \top)))^\alpha$ if and only if $\mathcal{A} \in (c_0, \ell_1)$ or $\mathcal{A} \in (c, \ell_1)$. Hence, by Lemma 2.2, it is deduced that

$$\sup_{N,M \in \mathcal{N}} \left| \sum_{i \in N} \sum_{j \in M} \frac{\varphi(j)}{\top(i)} a_i \right| < \infty.$$

Thus, it is concluded that $A = (c_0(\Delta(\varphi, \top)))^\alpha = (c(\Delta(\varphi, \top)))^\alpha$.□

By applying Theorem 2.3, the $\beta$- and $\gamma$-duals of the spaces $c_0(\Delta(\varphi, \top))$ and $c(\Delta(\varphi, \top))$ are established in the following theorem.

**Theorem 4.2.** *Define the sets $A_1$, $A_2$ and $A_3$ as follows:*

$$A_1 = \left\{ a = (a_i) \in \omega : \sup_{i \in \mathbb{N}} \sum_j \left| \sum_{l=j}^{i} a_l \frac{\varphi(j)}{\top(l)} \right| < \infty \right\},$$

$$A_2 = \left\{ a = (a_i) \in \omega : \lim_{i \to \infty} \sum_{l=j}^{i} a_l \frac{\varphi(j)}{\top(l)} \text{ exists for each } j \in \mathbb{N} \right\}$$

*and*

$$A_3 = \left\{ a = (a_i) \in \omega : \lim_{i \to \infty} \sum_j \sum_{l=j}^{i} a_l \frac{\varphi(j)}{\top(l)} \text{ exists} \right\}.$$

*Then,*$(c_0(\Delta(\varphi, \top)))^\beta = A_1 \cap A_2$ and $(c(\Delta(\varphi, \top)))^\beta = A_1 \cap A_2 \cap A_3$.

*Proof.* Let $a = (a_i) \in \omega$ and $\mathcal{B} = (b_{ij})$ be an infinite matrix with terms

$$b_{ij} = \begin{cases} \sum_{l=j}^{i} a_l \frac{\varphi(j)}{\top(l)} & , \quad \text{if } 1 \le j \le i \\ 0 & , \quad \text{if } j > i. \end{cases}$$

Hence, it is obtained that

$$\sum_{j=1}^{i} a_j x_j = \sum_{j=1}^{i} a_j \left( \sum_{l=1}^{j} \frac{\varphi(l)}{\top(j)} y_l \right) = \sum_{j=1}^{i} \left( \sum_{l=j}^{i} a_l \frac{\varphi(j)}{\top(l)} \right) y_j = \mathcal{B}_i y$$

for any $x = (x_i) \in c_0(\Delta(\varphi, \top))$ or $x = (x_i) \in c(\Delta(\varphi, \top))$. It follows that $ax \in cs$ for $x \in c_0(\Delta(\varphi, \top))$ or $x \in c(\Delta(\varphi, \top))$ if and only if $\mathcal{B}y \in c$ for $y \in c_0$ or $y \in c$. That is, $a \in (c_0(\Delta(\varphi, \top)))^\beta$ or $a \in (c(\Delta(\varphi, \top)))^\beta$ if and only if $\mathcal{B} \in (c_0, c)$ or $\mathcal{B} \in (c, c)$, respectively. Hence, by Lemma 2.2, it is concluded that $(c_0(\Delta(\varphi, \top)))^\beta = A_1 \cap A_2$ and $(c(\Delta(\varphi, \top)))^\beta = A_1 \cap A_2 \cap A_3$.

The gamma space can be proved in a similar way.

## 5. Certain matrix transformations

In this section, characterization of certain classes of matrices is given.

The following result is required to characterize the classes of matrices from $c_0(\Delta(\varphi, \top))$ and $c(\Delta(\varphi, \top))$ into $\ell_1, c_0, c, \ell_\infty$.

**Theorem 5.1.** *Let $\mathcal{X} = c_0$ or $\mathcal{X} = c$ and $\mathcal{Y}$ be an arbitrary subset of $\omega$. Then, $\mathcal{A} = (a_{ij}) \in (\mathcal{X}_{\Delta(\varphi,\top)}, \mathcal{Y})$ if andonly if$\mathcal{E}^{(i)} = \left(e_{lj}^{(i)}\right) \in (\mathcal{X}, c)$for each fixed$i \in \mathbb{N}$and$\mathcal{E} = (e_{ij}) \in (\mathcal{X}, \mathcal{Y})$,*

*where*

$$e_{lj}^{(i)} = \begin{cases} \sum_{k=j}^{l} a_{ik} \frac{\varphi(j)}{\top(k)} & , \quad 1 \le j \le l \\ 0 & , \quad j > l \end{cases}$$

*and*

$$e_{ij} = \sum_{k=j}^{\infty} a_{ik} \frac{\varphi(j)}{\top(k)}.$$

*Proof.* Let $\mathcal{A} \in (\mathcal{X}_{\Delta(\varphi,\top)}, \mathcal{Y})$ and $x \in \mathcal{X}_{\Delta(\varphi,\top)}$. Then, the equality

$$\sum_{j=1}^{l} a_{ij} x_j = \sum_{j=1}^{l} a_{ij} \left( \sum_{l=1}^{j} \frac{\varphi(l)}{\top(j)} y_l \right)$$
$$= \sum_{j=1}^{l} \left( \sum_{k=j}^{l} a_{ik} \frac{\varphi(j)}{\top(k)} \right) y_j = \sum_{j=1}^{l} e_{lj}^{(i)} y_j \tag{6}$$

holds. Since $\mathcal{A}x$ exists, it follows that $\mathcal{E}^{(i)} \in (\mathcal{X}, c)$ for each fixed $i \in \mathbb{N}$. It is deduced that $\mathcal{A}x = \mathcal{E}y$ as $l \to \infty$ in (6). Hence, $\mathcal{A}x \in \mathcal{Y}$ implies that $\mathcal{E}y \in \mathcal{Y}$; that is $\mathcal{E} \in (\mathcal{X}, \mathcal{Y})$.

Conversely, suppose that $\mathcal{E}^{(i)} = \left(e_{lj}^{(i)}\right) \in (\mathcal{X}, c)$ for each fixed $i \in \mathbb{N}$and $\mathcal{E} = (e_{ij}) \in (\mathcal{X}, \mathcal{Y})$. Let $x \in \mathcal{X}_{\Delta(\varphi,\top)}$. Then, $(e_{ij}) \in \mathcal{X}^{\beta}$ for each fixed $i \in \mathbb{N}$ implies that $(a_{ij}) \in (\mathcal{X}_{\Delta(\varphi,\top)})^{\beta}$for each fixed $i \in \mathbb{N}$. Hence, $\mathcal{A}x$ exists. From equality (6), it follows that $\mathcal{A}x = \mathcal{E}y$ as $l \to \infty$. This proves that $\mathcal{A} \in (\mathcal{X}_{\Delta(\varphi,\top)}, \mathcal{Y})$.

**Theorem 5.2.** *Let $\mathcal{A} = (a_{ij})$ be an infinite matrix. Then, the following statements hold:*

1. $\mathcal{A} \in (c_0(\Delta(\varphi,\top)), \ell_1)$ *if and only if*

$$\sup_{l\in\mathbb{N}} \sum_{j=1}^{l} \left| \sum_{k=j}^{l} a_{ik} \frac{\varphi(j)}{\top(k)} \right| < \infty \text{ for each fixed } i \in \mathbb{N}, \tag{7}$$

$$\lim_{l\to\infty} \sum_{k=j}^{l} a_{ik} \frac{\varphi(j)}{\top(k)} \text{ exists for each fixed } i, j \in \mathbb{N} \tag{8}$$

*and*

$$\sup_{N,M\in\mathcal{N}} \left| \sum_{i\in N} \sum_{j\in M} \sum_{k=j}^{\infty} a_{ik} \frac{\varphi(j)}{\top(k)} \right| < \infty. \tag{9}$$

2. $\mathcal{A} \in (c_0(\Delta(\varphi,\top)), c_0)$ *if and only if* (7), (8),

$$\sup_{i\in\mathbb{N}} \sum_{j} \left| \sum_{k=j}^{\infty} a_{ik} \frac{\varphi(j)}{\top(k)} \right| < \infty \tag{10}$$

*and*

$$\lim_{i\to\infty} \sum_{k=j}^{\infty} a_{ik} \frac{\varphi(j)}{\top(k)} = 0 \text{ for each } j \in \mathbb{N}. \tag{11}$$

3. $\mathcal{A} \in (c_0(\Delta(\varphi,\top)), c)$ *if and only if* (7), (8), (10) *and*

$$\lim_{i\to\infty} \sum_{k=j}^{\infty} a_{ik} \frac{\varphi(j)}{\top(k)} \text{ exists for each } j \in \mathbb{N}. \tag{12}$$

4. $\mathcal{A} \in (c_0(\Delta(\varphi,\top)), \ell_\infty)$ *if and only if* (7), (8) *and* (10).

*Proof.* The proof follows from Lemma 2.2 and Theorem 5.1.

**Theorem 5.3.** *Let $\mathcal{A} = (a_{ij})$ be an infinite matrix. Then, the following statements hold:*

1. $\mathcal{A} \in (c(\Delta(\varphi,\top)),\ell_1)$ *if and only if* (7), (8),

$$\lim_{l\to\infty}\sum_{j=1}^{l}\sum_{k=j}^{l} a_{ik}\frac{\varphi(j)}{\top(k)} \text{ exists for each } i \in \mathbb{N} \tag{13}$$

*and*(9).

2. $\mathcal{A} \in (c(\Delta(\varphi,\top)),c_0)$ *if and only if* (7), (8), (13), (11) *and*

$$\lim_{i\to\infty}\sum_{j}\sum_{k=j}^{\infty} a_{ik}\frac{\varphi(j)}{\top(k)} = 0. \tag{14}$$

3. $\mathcal{A} \in (c(\Delta(\varphi,\top)),c)$ *if and only if* (7), (8), (13), (10), (12) *and*

$$\lim_{i\to\infty}\sum_{j}\sum_{k=j}^{\infty} a_{ik}\frac{\varphi(j)}{\top(k)} \text{ exists.} \tag{15}$$

4. $\mathcal{A} \in (c(\Delta(\varphi,\top)),\ell_\infty)$ *if and only if* (7), (8), (13) *and* (10).

*Proof.* The proof follows from Lemma 2.2 and Theorem 5.1.

**Corollary 5.4.** *Let $\mathcal{A} = (a_{ij})$ be an infinite matrix. Then, the following statements hold:*

1. $\mathcal{A} \in (c_0(\Delta(\varphi,\top)),cs_0)$ *if and only if* (7), (8),

$$\sup_{i\in\mathbb{N}}\sum_{j}\left|\sum_{l=1}^{i}\sum_{k=j}^{\infty} a_{lk}\frac{\varphi(j)}{\top(k)}\right| < \infty \tag{16}$$

*and*

$$\lim_{i\to\infty}\sum_{l=1}^{i}\sum_{k=j}^{\infty} a_{lk}\frac{\varphi(j)}{\top(k)} = 0 \text{ for each } j \in \mathbb{N}. \tag{17}$$

2. $\mathcal{A} \in (c_0(\Delta(\varphi,\top)),cs)$ *if and only if* (7), (8), (16) *and*

$$\lim_{i\to\infty}\sum_{l=1}^{i}\sum_{k=j}^{\infty} a_{lk}\frac{\varphi(j)}{\top(k)} \text{ exists for each } j \in \mathbb{N}. \tag{18}$$

3. $\mathcal{A} \in (c_0(\Delta(\varphi,\top)),bs)$ *if and only if* (7), (8) *and* (16).

**Corollary 5.5.** *Let $\mathcal{A} = (a_{ij})$ be an infinite matrix. Then, the following statements hold:*

1. $\mathcal{A} \in (c(\Delta(\varphi, \top)), cs_0)$ *if and only if* (7), (8), (13), (16), (17) *and*

$$\lim_{i\to\infty} \sum_{j} \sum_{l=1}^{i} \sum_{k=j}^{\infty} a_{lk} \frac{\varphi(j)}{\top(k)} = 0. \tag{19}$$

2. $\mathcal{A} \in (c(\Delta(\varphi, \top)), cs)$ *if and only if* (7), (8), (13), (16), (18) *and*

$$\lim_{i\to\infty} \sum_{j} \sum_{l=1}^{i} \sum_{k=j}^{\infty} a_{lk} \frac{\varphi(j)}{\top(k)} \text{ exists.} \tag{20}$$

3. $\mathcal{A} \in (c(\Delta(\varphi, \top)), bs)$ *if and only if*(7), (8), (13) *and* (16).

The following resultis required to characterize the classes of matrices from $\ell_1, c_0, c, \ell_\infty$ into $c_0(\Delta(\varphi, \top))$ and $c(\Delta(\varphi, \top))$.

**Theorem 5.6.** *Let $\mathcal{X} = c_0$ or $\mathcal{X} = c$ and $\mathcal{Y}$ be an arbitrary subset of $\omega$.Given any infinite matrix $\mathcal{A} = (a_{ij})$, define the infinite matrix $\mathcal{D} = (d_{ij})$ as follows:*

$$d_{ij} = \sum_{k=i-1}^{i} (-1)^{i-k} \frac{\top(k)}{\varphi(i)} a_{kj}.$$

*Then, $\mathcal{A} = (a_{ij}) \in (\mathcal{Y}, \mathcal{X}_{\Delta(\varphi,\top)})$ if and only if*

$$\mathcal{D} = (d_{ij}) \in (\mathcal{Y}, \mathcal{X}).$$

*Proof.* Let $x \in \mathcal{Y}$. Then, the equality

$$\sum_{j=1}^{\infty} d_{ij} x_j = \sum_{j=1}^{\infty} \left( \sum_{k=i-1}^{i} (-1)^{i-k} \frac{\top(k)}{\varphi(i)} a_{kj} \right) x_j = \sum_{k=i-1}^{i} (-1)^{i-k} \frac{\top(k)}{\varphi(i)} \left( \sum_{j=1}^{\infty} a_{kj} x_j \right)$$

holds which means $\mathcal{D}_i(x) = \Delta(\varphi, \top)_i(\mathcal{A}x)$ for all $i \in \mathbb{N}$. That is, $\mathcal{D} = \Delta(\varphi, \top) \circ \mathcal{A}$ and so $\mathcal{A}x \in \mathcal{X}_{\Delta(\varphi,\top)}$ for any $x \in \mathcal{Y}$ if and only if $\mathcal{D}x \in \mathcal{X}$ for any $x \in \mathcal{Y}$.

**Theorem 5.7.** *Let $\mathcal{A} = (a_{ij})$ be an infinite matrix. Then, the following statements hold:*

1. $\mathcal{A} \in (\ell_1, c_0(\Delta(\varphi, \top)))$ *if and only if*

$$\lim_{i\to\infty} \sum_{k=i-1}^{i} (-1)^{i-k} \frac{\top(k)}{\varphi(i)} a_{kj} = 0 \text{ for each } j \in \mathbb{N} \tag{21}$$

*and*

$$\sup_{i,j\in\mathbb{N}}\left|\sum_{k=i-1}^{i}(-1)^{i-k}\frac{\top(k)}{\varphi(i)}a_{kj}\right|<\infty. \tag{22}$$

2. $\mathcal{A}\in(c_0,c_0(\Delta(\varphi,\top)))$ *if and only if* (21) *and*

$$\sup_{i\in\mathbb{N}}\sum_{j}\left|\sum_{k=i-1}^{i}(-1)^{i-k}\frac{\top(k)}{\varphi(i)}a_{kj}\right|<\infty. \tag{23}$$

3. $\mathcal{A}\in(c,c_0(\Delta(\varphi,\top)))$ *if and only if* (21) *and*

$$\lim_{i\to\infty}\sum_{j}\left(\sum_{k=i-1}^{i}(-1)^{i-k}\frac{\top(k)}{\varphi(i)}a_{kj}\right)=0. \tag{24}$$

4. $\mathcal{A}\in(\ell_\infty,c_0(\Delta(\varphi,\top)))$ *if and only if* (21) *and*

$$\lim_{i\to\infty}\sum_{j}\left|\sum_{k=i-1}^{i}(-1)^{i-k}\frac{\top(k)}{\varphi(i)}a_{kj}\right|=0. \tag{25}$$

*Proof.* The proof follows from Lemma 2.2 and Theorem 5.6.

**Theorem 5.8.** *Let $\mathcal{A}=(a_{ij})$ be an infinite matrix. Then, the following statements hold:*

1. $\mathcal{A}\in(\ell_1,c(\Delta(\varphi,\top)))$ *if and only if* (22)*and*

$$\lim_{i\to\infty}\sum_{k=i-1}^{i}(-1)^{i-k}\frac{\top(k)}{\varphi(i)}a_{kj}\text{ exists for each }j\in\mathbb{N}. \tag{26}$$

2. $\mathcal{A}\in(c_0,c(\Delta(\varphi,\top)))$ *if and only if*(23) *and* (26).
3. $\mathcal{A}\in(c,c(\Delta(\varphi,\top)))$ *if and only if* (23), (26) *and*

$$\lim_{i\to\infty}\sum_{j}\left(\sum_{k=i-1}^{i}(-1)^{i-k}\frac{\top(k)}{\varphi(i)}a_{kj}\right)\text{ exists.} \tag{27}$$

4. $\mathcal{A}\in(\ell_\infty,c(\Delta(\varphi,\top)))$ *if and only if* (26) *and*

$$\lim_{i\to\infty}\sum_{j}\left|\sum_{k=i-1}^{i}(-1)^{i-k}\frac{\top(k)}{\varphi(i)}a_{kj}\right|=\sum_{j}\left|\lim_{i\to\infty}\sum_{k=i-1}^{i}(-1)^{i-k}\frac{\top(k)}{\varphi(i)}a_{kj}\right|. \tag{28}$$

*Proof.* The proof follows from Lemma 2.2 and Theorem 5.6.

## 6. Compact operators on the space $c_0(\Delta(\varphi, \top))$

Now, some results are given to use in the sequel.

**Lemma 6.1.** *Let $a = (a_j) \in (c_0(\Delta(\varphi, \top)))^{\beta}$. Then, for all $x = (x_j) \in c_0(\Delta(\varphi, \top))$*

$$\sum_j a_j x_j = \sum_j \tilde{a}_j y_j \tag{29}$$

*and $\tilde{a} = (\tilde{a}_j) \in \ell_1$,*

*where*

$$\tilde{a}_j = \sum_{k=j}^{\infty} \frac{\varphi(j)}{\top(k)} a_k (j \in \mathbb{N}). \tag{30}$$

**Lemma 6.2.** *Let $a = (a_j) \in (c_0(\Delta(\varphi, \top)))^{\beta}$. Then, $\|a\|^*_{c_0(\Delta(\varphi,\top))} = \sum_j |\tilde{a}_j| < \infty$, where $\tilde{a} = (\tilde{a}_j)$ is the sequence with terms given by*(30).

*Proof.* If $a = (a_j) \in (c_0(\Delta(\varphi, \top)))^{\beta}$, it follows from Lemma 6.1 that $\tilde{a} = (\tilde{a}_j) \in \ell_1$ and the equality (29) holds. The equality $\|x\|_{\Delta(\varphi,\top)} = \|y\|_{\ell_\infty}$ implies that $x \in B_{c_0(\Delta(\varphi,\top))}$ if and only if $y \in B_{c_0}$. Hence, it is deduced that

$$\|a\|^*_{c_0(\Delta(\varphi,\top))} = \sup_{x \in B_{c_0(\Delta(\varphi,\top))}} \left|\sum_j a_j x_j\right| = \sup_{y \in B_{c_0}} \left|\sum_j \tilde{a}_j y_j\right| = \|\tilde{a}\|^*_{c_0}.$$

By Lemma 2.5, it is concluded that

$$\|a\|^*_{c_0(\Delta(\varphi,\top))} = \|\tilde{a}\|^*_{c_0} = \|\tilde{a}\|_{\ell_1} = \sum_j |\tilde{a}_j| < \infty.$$

In the rest of the paper, the matrix $\tilde{\mathcal{A}} = (\tilde{a}_{ij})$ is defined by an infinite matrix $\mathcal{A} = (a_{ij})$ such that

$$\tilde{a}_{ij} = \sum_{k=j}^{\infty} \frac{\varphi(j)}{\top(k)}$$

provided that the infinite sum is convergent.

**Lemma 6.3.** *Let $\mathcal{Y}$ be an arbitrary subset of $\omega$ and $\mathcal{A} = (a_{ij})$ be an infinite matrix. If $\mathcal{A} \in (c_0(\Delta(\varphi, \top)), \mathcal{Y})$, then $\tilde{\mathcal{A}} \in (c_0, \mathcal{Y})$ and $\mathcal{A}x = \tilde{\mathcal{A}}y$ for all $x \in c_0(\Delta(\varphi, \top))$.*

*Proof.* It follows from Lemma 6.1.

**Lemma 6.4.** *Let $\mathcal{Y} \in \{c_0, c, \ell_\infty\}$. If $\mathcal{A} \in (c_0(\Delta(\varphi, \top)), \mathcal{Y})$, then*

$$\|\mathcal{L}_{\mathcal{A}}\| = \|\mathcal{A}\|_{(c_0(\Delta(\varphi,\top)),\mathcal{Y})} = \sup_i \left(\sum_j |\tilde{a}_{ij}|\right) < \infty$$

*holds.*

**Theorem 6.5.**

1. *If* $\mathcal{A} \in (c_0(\Delta(\varphi, \top)), \ell_\infty)$, *then* $0 \leq \|\mathcal{L}_{\mathcal{A}}\|_\chi \leq \limsup_i \sum_j |\tilde{a}_{ij}|$ *holds.*
2. *If* $\mathcal{A} \in (c_0(\Delta(\varphi, \top)), c)$, *then*

$$\frac{1}{2}\limsup_i \sum_j |\tilde{a}_{ij} - \tilde{a}_j| \leq \|\mathcal{L}_{\mathcal{A}}\|_\chi \leq \limsup_i \sum_j |\tilde{a}_{ij} - \tilde{a}_j|$$

*holds.*

3. *If* $\mathcal{A} \in (c_0(\Delta(\varphi, \top)), c_0)$, *then*

$$\|\mathcal{L}_{\mathcal{A}}\|_\chi = \limsup_i \sum_j |\tilde{a}_{ij}|$$

*holds.*

4. *If* $\mathcal{A} \in (c_0(\Delta(\varphi, \top)), \ell_1)$, *then*

$$\lim_r \|\mathcal{A}\|^{(r)}_{(c_0(\Delta(\varphi,\top)),\ell_1)} \leq \|\mathcal{L}_{\mathcal{A}}\|_\chi \leq 4\lim_r \|\mathcal{A}\|^{(r)}_{(c_0(\Delta(\varphi,\top)),\ell_1)}$$

*holds, where* $\|\mathcal{A}\|^{(r)}_{(c_0(\Delta(\varphi,\top)),\ell_1)} = \sup_{N\in\mathcal{N}_r}\left(\sum_j \left|\sum_{i\in N} \tilde{a}_{ij}\right|\right)(r \in \mathbb{N})$.

*Proof.* (1) Let $\mathcal{A} \in (c_0(\Delta(\varphi, \top)), \ell_\infty)$. Since the series $\sum_{j=1}^{\infty} a_{ij}x_j$ converges for each $i \in \mathbb{N}$, it follows that $\mathcal{A}_i \in (c_0(\Delta(\varphi, \top)))^\beta$. Hence, Lemma 6.2 yields

$$\|\mathcal{A}_i\|^*_{c_0(\Delta(\varphi,\top))} = \|\tilde{\mathcal{A}}_i\|^*_{c_0} = \|\tilde{\mathcal{A}}_i\|_{\ell_1} = \left(\sum_j |\tilde{a}_{ij}|\right)$$

for each $i \in \mathbb{N}$. By using Lemma 2.9 (a), it is concluded that

$$0 \leq \|\mathcal{L}_\Lambda\|_\chi \leq \limsup_i \left(\sum_j |\tilde{a}_{ij}|\right).$$

(2) Let $\mathcal{A} \in (c_0(\Delta(\varphi, \top)), c)$. Then, Lemma 6.3 yields $\tilde{\mathcal{A}} \in (c_0, c)$. Hence, from Lemma 2.9 (c), it follows that

$$\frac{1}{2}\limsup_i \|\tilde{\mathcal{A}}_i - \tilde{a}\|^*_{c_0} \leq \|\mathcal{L}_\Lambda\|_\chi \leq \limsup_i \|\tilde{\mathcal{A}}_i - \tilde{a}\|^*_{c_0},$$

where $\tilde{a} = (\tilde{a}_j)$ and $\tilde{a}_j = \lim_i \tilde{a}_{ij}$ for each $j \in \mathbb{N}$.Moreover, Lemma 2.5 implies that

$$\|\tilde{\mathcal{A}}_i - \tilde{a}\|^*_{c_0} = \|\tilde{\mathcal{A}}_i - \tilde{a}\|_{\ell_1} = \left(\sum_j |\tilde{a}_{ij} - \tilde{a}_j|\right)$$

for each $i \in \mathbb{N}$. This completes the proof.

(3) Let $\mathcal{A} \in (c_0(\Delta(\varphi, \top)), c_0)$.Since

$$\|\mathcal{A}_i\|^*_{c_0(\Delta(\varphi,\top))} = \|\tilde{\mathcal{A}}_i\|^*_{c_0} = \|\tilde{\mathcal{A}}_i\|_{\ell_1} = \left(\sum_j |\tilde{a}_{ij}|\right)$$

holds for each $i \in \mathbb{N}$, it is concluded from Lemma 2.9 (b) that

$$\|\mathcal{L}_{\mathcal{A}}\|_{\chi} = \lim_i \sup \left( \sum_j |\tilde{a}_{ij}| \right).$$

(4)Let $\mathcal{A} \in (c_0(\Delta(\varphi, \top)), \ell_1)$.Then, Lemma 6.3 yields $\tilde{\mathcal{A}} \in (c_0, \ell_1)$.It follows from Lemma 2.10 that

$$\lim_r \left( \sup_{N \in \mathcal{N}_r} \left\| \sum_{i \in N} \tilde{\mathcal{A}}_i \right\|_{c_0}^{*} \right) \leq \|\mathcal{L}_{\mathcal{A}}\|_{\chi} \leq 4 \lim_r \left( \sup_{N \in \mathcal{N}_r} \left\| \sum_{i \in N} \tilde{\mathcal{A}}_i \right\|_{c_0}^{*} \right).$$

Moreover, Lemma 2.5 implies that

$$\left\| \sum_{i \in N} \tilde{\mathcal{A}}_i \right\|_{c_0}^{*} = \left\| \sum_{i \in N} \tilde{\mathcal{A}}_i \right\|_{\ell_1} = \left( \sum_j \left| \sum_{i \in N} \tilde{a}_{ij} \right| \right)$$

which completes the proof.

By combining Theorem 6.3 with Lemma 2.9 and Lemma 2.10, we have the following result.

**Corollary 6.6.**

1. *$\mathcal{L}_{\mathcal{A}}$ is compact for $\mathcal{A} \in (c_0(\Delta(\varphi, \top)), \ell_\infty)$ if*

$$\lim_i \sum_j |\tilde{a}_{ij}| = 0.$$

2. *$\mathcal{L}_{\mathcal{A}}$ is compact for $\mathcal{A} \in (c_0(\Delta(\varphi, \top)), c)$ if and only if*

$$\lim_i \sum_j |\tilde{a}_{ij} - \tilde{a}_j| = 0.$$

3. *$\mathcal{L}_{\mathcal{A}}$ is compact for $\mathcal{A} \in (c_0(\Delta(\varphi, \top)), c_0)$ if and only if*

$$\lim_i \sum_j |\tilde{a}_{ij}| = 0.$$

4. *$\mathcal{L}_{\mathcal{A}}$ is compact for $\mathcal{A} \in (c_0(\Delta(\varphi, \top)), \ell_1)$ if and only if*

$$\lim_r \|\mathcal{A}\|_{(c_0(\Delta(\varphi, \top)), \ell_1)}^{(r)} = 0,$$

*where* $\|\mathcal{A}\|_{(c_0(\Delta(\varphi, \top)), \ell_1)}^{(r)} = \sup_{N \in \mathcal{N}_r} \left( \sum_j \left| \sum_{i \in N} \tilde{a}_{ij} \right| \right)$.

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