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Research Article

On Arithmetic Continuity in Cone Metric Space

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Abstract: William Henry Ruckle introduced the notion of arithmetic convergence in the sense that a sequence g defined on the set of natural numbers $\mathbb{N}$ is said to be arithmetic convergent if for each $\varepsilon > 0$ there is an integer n such that for every integer m, $|g(m) - g(\langle m, n\rangle)| < \varepsilon$, where $\langle m, n\rangle$ denotes the greatest common divisor of m and n. In this paper, the notion of arithmetic convergence has been extended to cone metric space. Using the concept of arithmetic convergence, arithmetic continuity and arithmetic compactness have been defined in cone metric spaces and give some interesting results.

Keywords: Cone metric space, arithmetic convergence, arithmetic continuous, arithmetic compactness.

AMS Mathematics Subject Classification: Primary: 40A05; Secondary: 26A15.

I. Introduction

The concept of cone metric spaces is a very recent and interesting generalization of usual metric space where the set of real numbers is replaced by an ordered Banach space. After the initial introduction of this space by Huang and Zhang (2007), a lot of work has been done on this structure, especially on fixed point theory. In this paper, we focus on a different direction and study arithmetic convergence, arithmetic continuity and related concepts in cone metric space. However, it should be noted that due to the absence of real numbers (which is replaced by an ordered Banach space), the methods of proofs are not always analogous to the usual metric case.

Ruckle (2012), introduced the notion *arithmetic convergence* as a sequence $x = (x_m)$ defined on the set of natural numbers $\mathbb{N}$ is said to be arithmetic convergent if for each $\varepsilon > 0$ there is an integer n such that for every integer m, we have $\left|x_m - x_{\langle m,n\rangle}\right| < \varepsilon$. Here $\langle m, n\rangle$ denotes the greatest common divisor of m and n. For details on arithmetic convergence and arithmetic continuity, we refer to (Yaying and Hazarika, 2017a, 2017b, 2017c, 2018).

In this article, we first introduce the concept of arithmetic convergence and using this notion we define arithmetic continuity in a cone metric space and prove some interesting results. Finally, we introduce arithmetic compactness and give some interesting results.

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II. Cone Metric Spaces

Let us recall some definitions and results on cone metric spaces which can be found in (Huang and Zhang, 2007).

Definition 1. *A subset P of a real Banach space E is called a cone if and only if*

(1) P is closed, nonempty and $P \neq \{\theta\}$;

(2) If $a, b \in \mathbb{R}^+$ and $x, y \in P$, then $ax + by \in P$;

(3) If both $x \in P$ and $-x \in P$, then $x = \theta$.

Given a cone P in E, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x = y$, while $x \ll y$ will stand for $y - x \in int\, P$ (interior of P).

Throughout the text, we always suppose that E is a Banach space, P is a solid cone i.e. $int\, P \neq \phi$ in E and $\leq$ is a partial ordering with respect to P.

Definition 2. *A cone metric space is an ordered pair (X, d), where X is a non-empty set and $d: X \times X \to E$ is a mapping satisfying:*

(i) $\theta \leq d(x, y)$ *for all* $x, y \in X$;

(ii) $d(x, y) = \theta$ *if and only if* $x = y$;

(iii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iv) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Example 1. *Let $E = \mathbb{R}^2, P = \{(x, y) \in E: x, y \geq 0\} \subset \mathbb{R}^2$ and $d: X \times X \to E$ be such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant and $x, y \in X$. Then it is well known that (X, d) is a cone metric space.*

Definition 3 (Convergence)**.** *A sequence $(x_m)_{n\in\mathbb{N}}$ converges to $x_0 \in X$ if for every $c \in E$ with $\theta \ll c$ (i.e. $c - \theta \in int\, P$) there exists a natural number N such that $d(x_m, x_0) \ll c \;\; \forall m \geq N$.*
We denote this by writing $x_m \to x_0$.

Definition 4 (Cauchyness)**.** *Let (X, d) be a cone metric space. We say that a sequence $(x_m)_{m\in\mathbb{N}}$ is cauchy if for every $c \in E$ with $\theta \ll c$ there exists $N \in \mathbb{N}$ such that for $m \geq n \geq N$, $d(x_n, x_m) \ll c$ holds.*

Definition 5 (Sequencial Compactness)**.** *A set $S \subset X$ is sequencially compact if every sequence in X has a convergent subsequence.*

Definition 6. *Let* (X, d) *be a cone metric space and* $f: X \to E$. *Then, the function* f *is called uniformly continuous on* X *if for* $\varepsilon > 0$ *and* $c \in E$ *with* $\theta \ll c$

$$d(x, y) \ll c \Rightarrow |f(x) - f(y)| < \varepsilon \ \forall x, y \in X.$$

Lemma 2.1 (Radenovic and Kadelburg, 2011)

(1) If $u \leq v$ *and* $v \ll w$, *then* $u \ll w$.

(2) If $u \ll v$ *and* $v \leq w$, *then* $u \ll w$.

(3) If $u \ll v$ *and* $v \ll w$, *then* $u \ll w$.

(4) If $\theta \leq u \ll c$ *for each* $c \in int\ P$ *then* $u = \theta$.

III. Arithmetic continuity in cone metric space

In this section we introduce the concept of *arithmetic continuity* and *arithmetic compactness* in a cone metric space and establish some interesting results related to these notions.

Definition 7. *A sequence* $x = (x_m)$ *is called arithmetically convergent in a cone metric space* (X, d) *if for every* $c \in E$ *with* $\theta \ll c$ $(i.e.\ c - \theta \in int\ P)$ *there is an integer* n *such that for every integer* m, *we have,* $d(x_m, x_{\langle m,n \rangle}) \ll c$.

Definition 8. *Let* (X, d_X) *and* (Y, d_Y) *be two cone metric spaces. A function* $f : X \to Y$ *is arithmetic continuous if it transforms arithmetic convergent sequences in* X *to arithmetic convergent sequences in* Y. *In other words, the sequence* (x_n) *is arithmetic convergent implies the sequence* $(f(x_n))$ *is also arithmetic convergent.*

Theorem 3.1. *The composition of two arithmetic continuous functions in a cone metric space* (X, d) *is again arithmetic continuous.*

Proof. Let f and g be two arithmetic continuous functions. We have to prove that the function $f \circ g(x_m) = f(g(x_m))$ is arithmetic continuous function.

Let (x_m) be any arithmetic convergence sequence in the cone metric space X. Since g is arithmetic continuous, so the sequence $(g(x_m))$ is also arithmetic convergent. Furthermore, it is given that f is arithmetic continuous, hence it transforms arithmetic convergence sequence $(g(x_m))$ to arithmetic convergence sequence $(f(g(x_m)))$. Hence the result follows.

Definition 9. *A sequence of functions* (f_m) *from a cone metric space* (X, d_X) *to a cone metric space* (Y, d_Y) *is said to be arithmetic convergent if for* $c' \in E'$ *with* $\theta' \ll c'$ *and* $\forall\, x \in X$ *there exists an integer* n *such that for every integer* m

$$d_Y\left(f_m(x), f_{\langle m,n\rangle}(x)\right) \ll c'.$$

Note that, here we have taken d_Y as a cone metric from $Y \times Y$ to a real Banach space E' with a cone $P' \subset E'$ and θ' is zero in E'.

Theorem 3.2. *Let (X, d_X) and (Y, d_Y) be two cone metric spaces. If $f: X \to Y$ is uniformly continuous then it is arithmetic continuous.*

Proof. Let f be uniformly continuous and (x_m) be any arithmetic convergence sequence in X. Since f is uniformly continuous, for a given $c \in E$ with $\theta \ll c$ such that for every x, y with $d_X(x, y) \ll c, d_Y(f(x), f(y)) \ll c'$. Again, the sequence (x_m) is arithmetic convergent, hence for the same $c \in E$ there exists a positive integer n such that

$d_X(x_m, x_{\langle m,n\rangle}) \ll c$ for each $m \Rightarrow d_Y(f(x_m), f(x_{\langle m,n\rangle})) \ll c'$ for each m

$\Rightarrow$ the sequence $(f(x_m))$ is arithmetic convergent.

$\Rightarrow$ the function f is arithmetic continuous.

This completes the proof.

Theorem 3.3. *If (f_m) be a sequence of arithmetic continuous functions from a cone metric space (X, d_X) to a cone metric space (Y, d_Y) and (x_0) is a point in X such that*

$$\lim_{x \to x_0} f_m(x) = y_m,$$

then (y_m) is also arithmetic convergent.

Proof. Since the sequence (f_m) is arithmetic convergent, therefore, for $c' \in E'$ with $\theta' \ll c'$ and $\forall\, x \in X$ there exists an integer n such that for every integer m

$$d_Y(f_m(x), f_{\langle m,n\rangle}(x)) \ll c'.$$

Keeping n, m fixed and letting $x \to x_0$,

$$d_Y\left(y_m, y_{\langle m,n\rangle}\right) \ll c', \ \forall m.$$

Hence, the sequence (y_m) is arithmetic convergent.

Theorem 3.4. *If (f_m) is a sequence of arithmetic continuous functions from a cone metric space (X, d_X) to a cone metric space (Y, d_Y) and (f_m) converges uniformly to a function f, then f is arithmetic continuous.*

Proof. Let $c' \in E'$ with $\theta' \ll c'$ and (x_m) be any arithmetic convergent sequence in X.

Since, $f_m \to f$ uniformly, there exist a positive integer N such that

$$d_Y\big(f(x), f_m(x)\big) \ll \frac{c'}{3},\ \forall\, m \geq N \text{ and } x \in X. \tag{3.1}$$

In particular for $m = m_1$, we have

$$d_Y\left(f_{m_1}(x), f(x)\right) \ll \frac{c'}{3}. \tag{3.2}$$

Furthermore, (f_m) is given to be a sequence of arithmetic continuous functions, therefore,

$$d_Y\,(f_{m_1}(x_m), f_{m_1}\,(x_{\langle m,n\rangle}\,)) \ll \frac{c'}{3}, \tag{3.3}$$

Also, from (3.1),

$$d_Y\,\left(f\left(x_{\langle m,n\rangle}\right), f_{m_1}\left(x_{\langle m,n\rangle}\right)\right) \ll \frac{c'}{3}. \tag{3.4}$$

Thus, from (3.2), (3.3), (3.4)

$d_Y(f(x_{\langle m,n\rangle}), f(x_m))$
$= d_Y\,(f(x_{\langle m,n\rangle}), f_{m_1}(x_{\langle m,n\rangle})) + d_Y\,(f_{m_1}\,(x_{\langle m,n\rangle}), f_{m_1}\,(x_m)) + d_Y\,(f_{m_1}(x_m), f(x_m))$
$\ll c'$.

Thus, f transforms arithmetic convergent sequence in X to arithmetic convergent sequence in Y. Hence f is arithmetic continuous function.

Theorem 3.5. *The set of all arithmetic continuous functions from cone metric space (X, d_X) to cone metric space (Y, d_Y) is a closed subset of all continuous functions i.e. $AC(X,Y) = \overline{AC(X,Y)}$ where $AC(X,Y)$ is the set of all arithmetic continuous functions from cone metric space (X, d_X) to cone metric space (Y, d_Y) and $\overline{AC(X,Y)}$ denotes the closure of $AC(X,Y)$.*

Proof: Let $f \in \overline{AC(X,Y)}$. Then there exists a sequence of points in $AC(X,Y)$ such that $\lim f_m = f$. Let $c' \ll E'$ with $\theta' \ll c'$ and (x_m) be any arithmetic convergent sequence in X.

Since, $fm \to f$ uniformly, there exist a positive integer N such that

$$d_Y\big(f(x), f_m(x)\big) \ll \frac{c'}{3},\ \forall\, m \geq N \text{ and } x \in X. \tag{3.5}$$

In particular for $m = m_1$, we have

$$d_Y\left(f_{m_1}(x), f(x)\right) \ll \frac{c'}{3}. \tag{3.6}$$

Furthermore, (f_m) is given to be a sequence of arithmetic continuous functions, therefore,

$$d_Y\left(f_{m_1}(x_m), f_{m_1}\left(x_{\langle m,n\rangle}\right)\right) \ll \frac{c'}{3}, \tag{3.7}$$

Also, from (3.5),

$$d_Y(f(x_{\langle m,n\rangle}), f_{m_1}(x_{\langle m,n\rangle})) \ll \frac{c'}{3}. \tag{3.8}$$

Thus, from (3.6), (3.7), (3.8)

$$\begin{aligned} &d_Y(f(x_{\langle m,n\rangle}), f(x_m)) \\ &= d_Y(f(x_{\langle m,n\rangle}), f_{m_1}(x_{\langle m,n\rangle})) + d_Y(f_{m_1}(x_{\langle m,n\rangle}), f_{m_1}(x_m)) + d_Y(f_{m_1}(x_m), f(x_m)) \\ &\ll c'. \end{aligned}$$

Hence f is arithmetic continuous function. So $f \in AC(X, Y)$. This completes the proof.

Corollary 3.1. *The set of all arithmetic continuous functions from a cone metric space (X, d_X) to cone metric space (Y, d_Y) is a complete subspace of the space of all continuous functions.*

Arithmetic Compactness

Definition 10. *A subset A of a cone metric space (X, d) is called arithmetic compact if every sequence in A has arithmetic convergent subsequence.*

Theorem 4.1. *An arithmetic continuous image of an arithmetic compact subset of a cone metric space (X, d_X) is arithmetic compact.*

Proof. Let (X, d_X) and (Y, d_Y) be cone metric spaces. Let $f: X \to Y$ be arithmetic continuous function and $A \subset X$ be arithmetic compact. Let (y_m) be a sequence in $f(A)$. Then we can write $y_m = f(x_m)$ where $x_m \in X$ for each $m \in \mathbb{N}$. Since A is arithmetic compact, there exists a forward convergent subsequence (x_{m_k}) of (x_m). Again, it is given that f is arithmetic continuous, this implies that $f(x_{m_k})$ is arithmetic convergent subsequence of $f(x_m)$. Hence, $f(A)$ is arithmetic compact.

Theorem 4.2. *Any closed subset of an arithmetic compact subset of a cone metric space (X, d) is arithmetic compact.*

Proof. Let A be any arithmetic compact subset of X and B be a closed subset of A. Let $x = (x_m)$ be any sequence of points in B. Then $x = (x_m)$ is a sequence of points in A. Since A is arithmetic compact, therefore there exists an arithmetic convergent subsequence (x_{m_k}) of the sequence x. Since B is closed, so any sequence $x = (x_m)$ of points in B has arithmetic convergent subsequence in B.

Hence, the result.

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